

Sensitivity analysis in design of solids and structures

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- sensitivity analysis
- applications of sensitivity analysis
 - worst imperfection analysis
 - optimization
 - multi-scale modeling
 - FE^2
 - MIEL scheme
 - asymptotic methods
 - stochastic methods

Appreciation



dr.
Niko Kristanič



asist.dr.
Teja Melink



asist.dr.
Blaž Hudobivnik



Nina Zupan

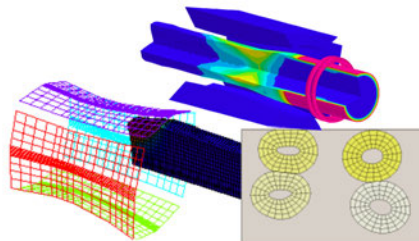
and ARRS, DFG, EU ...

Primal analysis - Sensitivity analysis

Primal analysis - response of the system

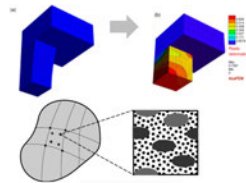
M5 problems

- multi-field
- multi-scale
- multi-body
- multi-phase
- multi-objective



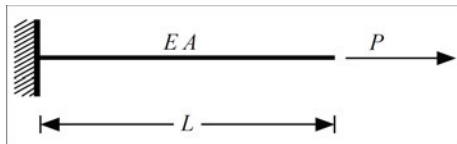
Sensitivity analysis - derivatives of the response Indispensable part of future computational algorithms

- imperfection analysis
- structural optimization procedures
- advanced path-following procedures
- efficient solution of coupled problems
- multi-scale methods
- efficient stochastic analysis (response



Sensitivity problem - differentiation of closed form solution

Example



- primal problem $\leadsto$ closed form solution of response

$$R = \frac{EA}{L}u - P = 0 \quad \leadsto \quad u = \frac{PL}{EA}$$

- sensitivity parameters

$$\Phi = \{A, E, L, P\}$$

- arbitrary order sensitivity of response with respect to parameter L

$$\frac{\partial u}{\partial L} = \frac{P}{EA}, \quad \frac{\partial^2 u}{\partial L^2} = 0$$

- arbitrary order sensitivity of response with respect to parameter A

$$\frac{\partial u}{\partial A} = -\frac{PL}{EA^2}, \quad \frac{\partial^2 u}{\partial A^2} = \frac{2PL}{EA^3}, \dots$$

- sensitivity problem needs solution of primal problem!

Generalization to an arbitrary number of unknowns

- Primal problem is defined by residual equation

$$\mathbf{R}(\mathbf{p}(\boldsymbol{\phi}), \boldsymbol{\phi}) = \mathbf{0}$$

- let assume that we have solved the primal problems using the Newton-Raphson iterative procedure

$$\mathbf{K}\Delta\mathbf{p}^{(i)} = -\mathbf{R}^{(i)}$$

- First order sensitivity: $\frac{D\mathbf{p}}{D\phi_i}$

- $\frac{\partial \mathbf{R}}{\partial \mathbf{p}} \frac{D\mathbf{p}}{D\phi_i} + \frac{\partial \mathbf{R}}{\partial \phi_i} = 0$

- $\frac{D\mathbf{p}}{D\phi_i}$ follows from the system of linear equations

$$\mathbf{K} \frac{D\mathbf{p}}{D\phi_i} = -{}^I\tilde{\mathbf{R}}$$

- $\mathbf{K}$ is already decomposed from primal analysis !

- ${}^I\tilde{\mathbf{R}}$ - first order pseudo load vector

$${}^I\hat{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial \phi_i}$$

Sensitivity problems - alternatives

- closed form solution is available only for very special cases
- Alternative numerical procedures:

A finite difference approximation

- inexact
- higher order sensitivities are ill conditioned

B Discrete derivatives

B.1 differentiation of discretized residual equation

$$\mathbf{R}(\mathbf{p}(\boldsymbol{\phi}), \boldsymbol{\phi}) = \mathbf{0}$$

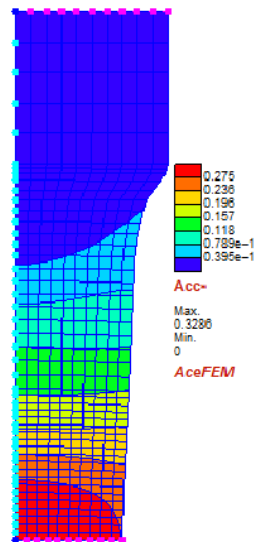
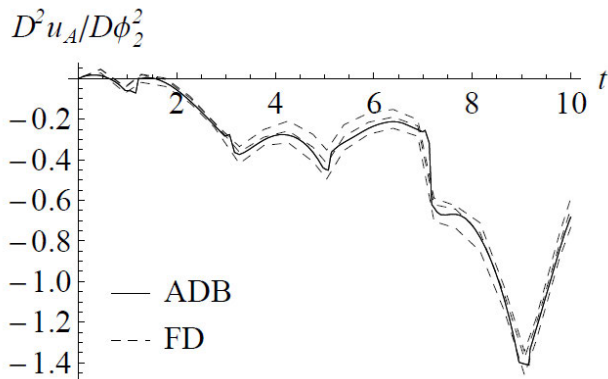
B.2 differentiation of complete solution procedure (e.g. Newton iterative procedure, explicit scheme,..)

$$\mathbf{K}\Delta\mathbf{p}^{(i)} = -\mathbf{R}^{(i)}$$

C Continuum derivatives

- differentiation of continuum equations
- discretization of sensitivity problem
- if the same mesh then the same results as discrete derivatives

FD higher order sensitivities are ill conditioned



Sensitivity analysis - second order direct differentiation

- Second order sensitivity: $\frac{D^2 \mathbf{p}}{D\phi_i D\phi_j}$
- Primal problem is defined by residual equation $\mathbf{R}(\mathbf{p}(\boldsymbol{\phi}), \boldsymbol{\phi}) = 0$
- $\frac{\partial \mathbf{R}}{\partial \mathbf{p}} \frac{D\mathbf{p}}{D\phi_i} + \frac{\partial \mathbf{R}}{\partial \phi_i} = 0 \leadsto$
 $\frac{\partial^2 \mathbf{R}}{\partial \mathbf{p}^2} \frac{D\mathbf{p}}{D\phi_i} \frac{D\mathbf{p}}{D\phi_j} + \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p} \partial \phi_j} \frac{D\mathbf{p}}{D\phi_i} + \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p} \partial \phi_i} \frac{D\mathbf{p}}{D\phi_j} + \frac{\partial \mathbf{R}}{\partial \mathbf{p}} \frac{D^2 \mathbf{p}}{D\phi_i D\phi_j} + \frac{\partial^2 \mathbf{R}}{\partial \phi_i \partial \phi_j} = 0$
- $\frac{D^2 \mathbf{p}}{D\phi_i D\phi_j}$ follows from the linear system $\mathbf{K} \frac{D^2 \mathbf{p}}{D\phi_i D\phi_j} = -{}^{II} \tilde{\mathbf{R}}$
- ${}^{II} \tilde{\mathbf{R}} = \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p}^2} \frac{D\mathbf{p}}{D\phi_i} \frac{D\mathbf{p}}{D\phi_j} + \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p} \partial \phi_j} \frac{D\mathbf{p}}{D\phi_i} + \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p} \partial \phi_i} \frac{D\mathbf{p}}{D\phi_j} + \frac{\partial^2 \mathbf{R}}{\partial \phi_i \partial \phi_j}$

Discrete and Continuum derivatives

Basics about discrete and continuum sensitivities:

- derivatives are analytically exact
- sensitivity problem is linear also for nonlinear problems
- no problems with the convergence
- we need sensitivity pseudo-load vectors

- $I \tilde{\mathbf{R}}$ - first order pseudo load vector

$$I \tilde{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial \phi_i}$$

- $II \tilde{\mathbf{R}}$ - second order pseudo load vector

$$II \tilde{\mathbf{R}} = \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p}^2} \frac{D\mathbf{p}}{D\phi_i} \frac{D\mathbf{p}}{D\phi_j} + \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p} \partial \phi_j} \frac{D\mathbf{p}}{D\phi_i} + \frac{\partial^2 \mathbf{R}}{\partial \mathbf{p} \partial \phi_i} \frac{D\mathbf{p}}{D\phi_j} + \frac{\partial^2 \mathbf{R}}{\partial \phi_i \partial \phi_j}$$

Discrete and Continuum derivatives

Basics about discrete and continuum sensitivities:

- in the case of standard FE procedure

$${}^I\tilde{\mathbf{R}} = \sum_{e=1}^{n_e} \sum_g w_{gp} {}^I\tilde{\mathbf{R}}_g$$

$${}^{II}\tilde{\mathbf{R}} = \sum_{e=1}^{n_e} \sum_g w_{gp} {}^{II}\tilde{\mathbf{R}}_g$$

- Gauss point contribution ${}^I\tilde{\mathbf{R}}_g, {}^{II}\tilde{\mathbf{R}}_g$ can be obtained:
 - analytically $\leadsto$ **DIRECT DIFFERENTIATION** or adjoint method
 - finite difference approximation $\leadsto$ **SEMI-ANALYTICAL** methods (ABAQUS)

Example: Transient coupled problems (plasticity) - $^{II}\tilde{\mathbf{R}}_g$

[illegible]

DIRECT DIFFERENTIATION methods

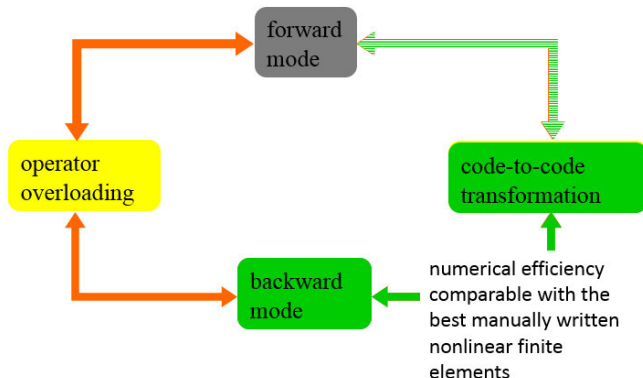
Necessary expressions can be derived:

- manually
- automatic differentiation
 - code-to-code transformation
 - difficult to implement
 - fast
 - expression growth for higher order derivatives
 - operator overloading (interpreter of the code is added)
 - easy to implement
 - slow and memory consuming
 - higher order derivatives easy to implement

Automatic differentiation and FEM

forward mode AD: $\frac{\text{cost}(y, \nabla y)}{\text{cost}(y)} \approx \alpha(n + 1)$

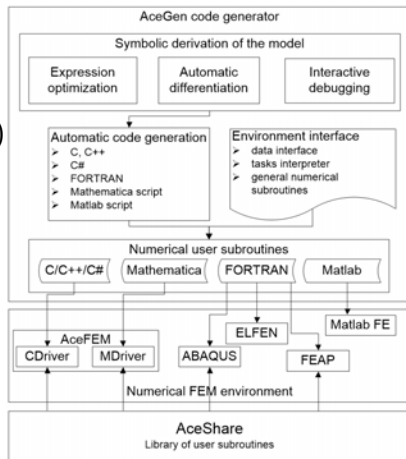
backward mode AD: $\frac{\text{cost}(y, \nabla y)}{\text{cost}(y)} < 5 \dots$ (Wolfe, 1982)



Hybrid symbolic–numeric approach (AceGen/AceFEM)

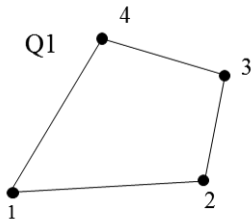
Essential components :

- ❶ Automatic differentiation based (ADB) formulation of problem
- ❷ Symbolic description (Mathematica)
- ❸ Expression swell control
 - Standard expression swell control
 - tensor algebra, matrix algebra
 - representative expressions
 - Automated expression swell control
 - introduction of intermediate variables
 - expression optimization
 - simultaneous code optimization
- ❹ Interactive debugging of automatically generated codes



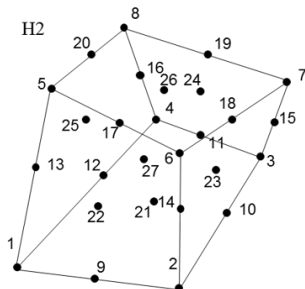
Sensitivity analysis - code complexity

Psychical size of the code for the evaluation of pseudo-load vectors



Q1 - elasto-plastic

Task	Code size	<i>AceGen</i>
$\mathbf{K}_e$ & $\mathbf{R}_e$	48 kB	55 s
$+^I \tilde{\mathbf{R}}_e$	112 kB	8 s
$+^{II} \tilde{\mathbf{R}}_e$	217 kB	17 s

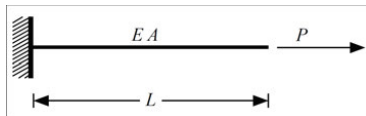


H2 - elasto-plastic

Task	Code size	<i>AceGen</i>
$\mathbf{K}_e$ & $\mathbf{R}_e$	136 kB	151 s
$+^I \tilde{\mathbf{R}}_e$	280 kB	400 s
$+^{II} \tilde{\mathbf{R}}_e$	621 kB	1717 s

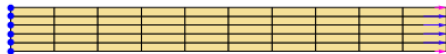
Design velocity field

Global level



global parameter
 L

Element level



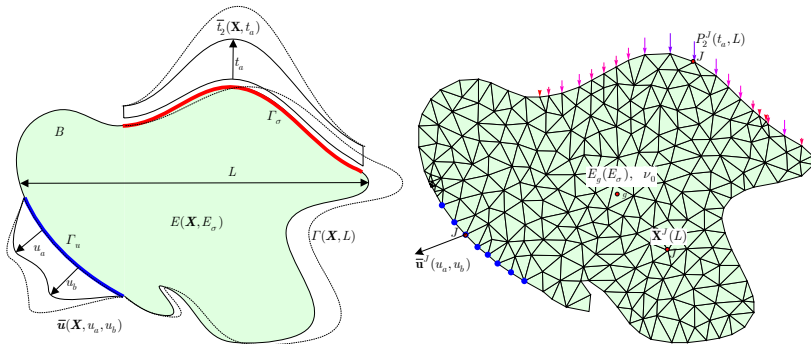
input data of element subroutines
nodal coordinates $\mathbf{X}(L)$

- The missing part in the chain rule between the global level and the element level is filled in by **design velocity field**

$$D_L \mathbf{X} = \frac{\partial \mathbf{X}}{\partial L}$$

- $D_L \mathbf{X}$ is input data of the sensitivity problem

Continuum problem vs Discretized problem



pseudo-load vector for parameter $\phi = L$

$$\frac{\partial \mathbf{R}}{\partial \phi} = \frac{\partial \mathbf{R}}{\partial L} = \mathbf{A} \sum_{e=1}^{n_e} \sum_{J=1}^{n_{en}} \left(\frac{\partial \mathbf{R}_e}{\partial X_1^J} \frac{DX_1^J}{DL} + \frac{\partial \mathbf{R}_e}{\partial X_2^J} \frac{DX_2^J}{DL} + \dots \right)$$

Design velocity matrix

- input data fields of dicretized problem

$$\boldsymbol{\psi} = \left\{ \left\{ \bar{u}_1^J, \bar{u}_2^J, \bar{P}_1^J, \bar{P}_2^J, X_1^J, X_2^J, E^J, \nu^J \right\} : J = 1, \dots, n_{tn} \right\}.$$

- design velocity matrix $D_{\Phi} \boldsymbol{\psi}^J$ of the problem

$$D_{\Phi} \boldsymbol{\psi}^J =$$

L	t_a	u_a	u_b	E_{σ}	ν_0	
0	0	$\frac{D\bar{u}_1^J}{Du_a}$	$\frac{D\bar{u}_1^J}{Du_b}$	0	0	u_1^J
0	0	$\frac{D\bar{u}_2^J}{Du_a}$	$\frac{D\bar{u}_2^J}{Du_b}$	0	0	u_2^J
0	0	0	0	0	0	P_1^J
$\frac{DP_2^J}{DL}$	$\frac{DP_2^J}{Dt_a}$	0	0	0	0	P_2^J
$\frac{DX_1^J}{DL}$	0	0	0	0	0	X_1^J
$\frac{DX_2^J}{DL}$	0	0	0	0	0	X_2^J
0	0	0	0	$\frac{DE^J}{DE_{\sigma}}$	0	E^J
0	0	0	0	0	$\frac{D\nu}{D\nu_0}$	ν^J

- worst imperfection analysis
- limit load optimisation (optimisation based design of structures)
- multi-scale modeling and sensitivity analysis
- stochastic analysis
- asymptotic numerical methods

Motivation - modern structural analysis

EUROCODE requires GMNIA FEM static analysis of structures

- Consideration of geometrical nonlinearity
- Consideration of material nonlinearity
- Consideration of initial imperfections

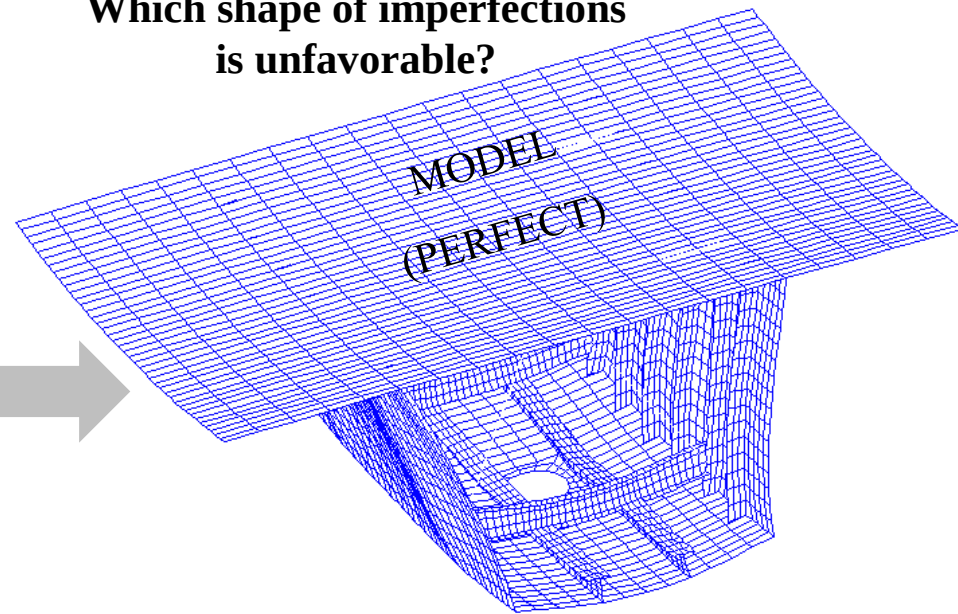
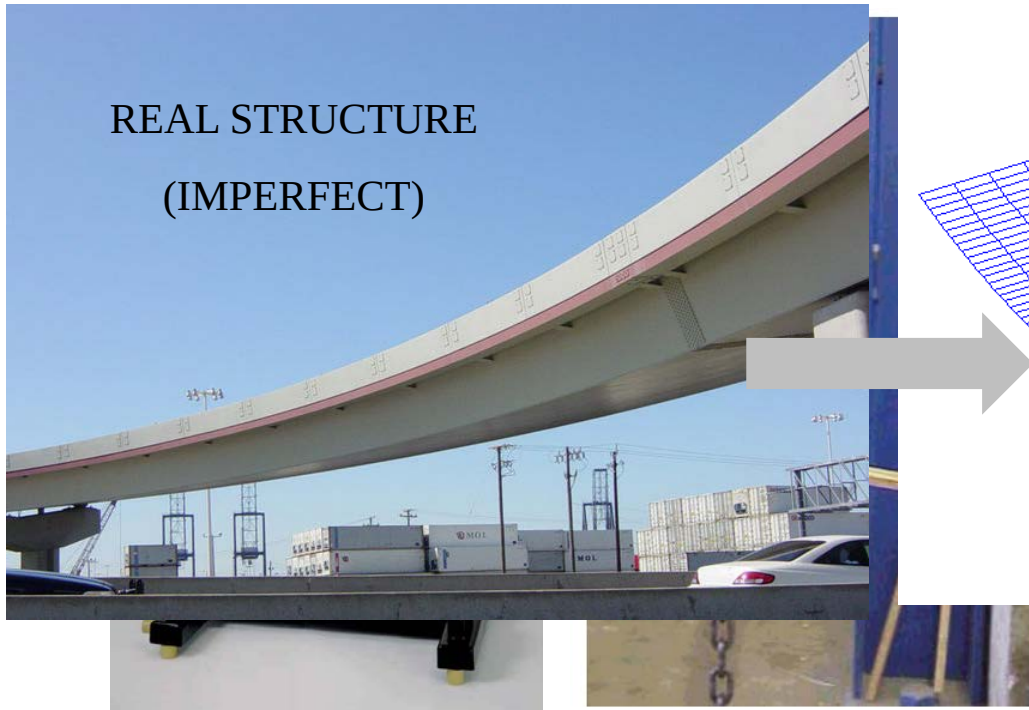
Flexural buckling

Lateral buckling

Local buckling
**Which shape of imperfections
is unfavorable?**

REAL STRUCTURE
(IMPERFECT)

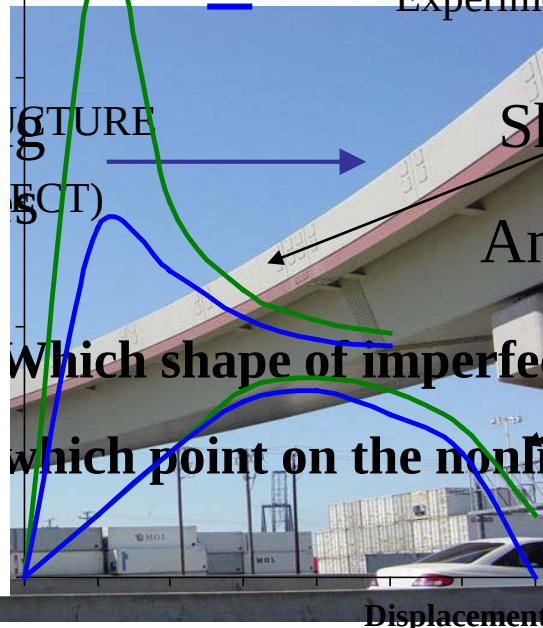
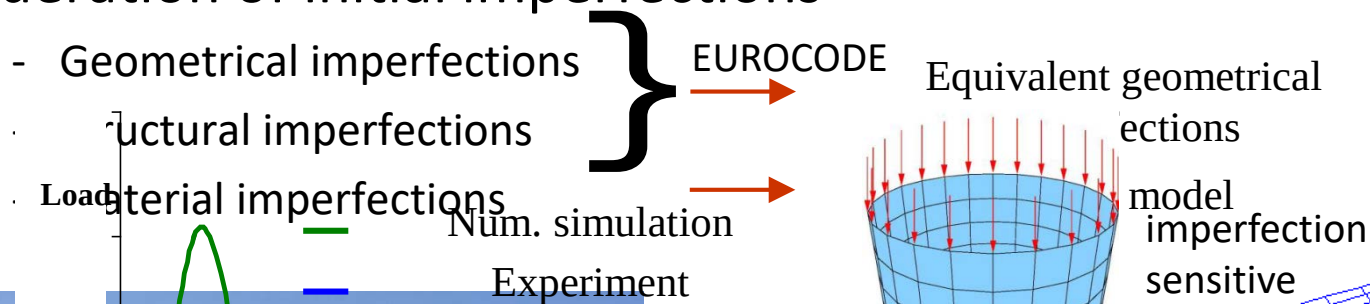
MODEL
(PERFECT)



Initial imperfections and nonlinear analyses

GMNIA FEM static analysis

- Consideration of geometrical nonlinearity
- Consideration of material nonlinearity
- Consideration of initial imperfections



Shape of imperfections

Amplitude of imperfections

Which shape of imperfections is realistic?

Which point on the nonlinear load-displacement path?

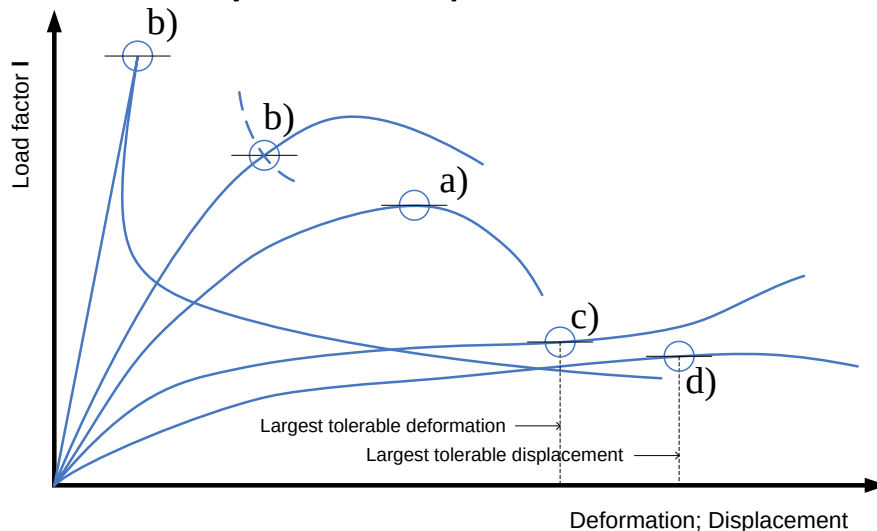


is UNKNOWN
 (PERFECT)
 tions is limited.

able?

Imperfection analysis

- “perturbation approach” (Koiter 1945)
 - stability analysis of ideal structure
- “direct approach”
 - limit load analysis of imperfect structure



- find the most unfavorable imperfection
- Deml and Wunderlich 1997, Lund 2010, elastic stability
- Kristanič and Korelc 2008, fully nonlinear approach
- "stochastic approach"



“direct approach”

Concept of most unfavorable imperfection:

- $\mathbf{X}_p$ - perfect geometry
- $\mathbf{X} = \mathbf{X}_p + \bar{\mathbf{X}}$ - imperfect geometry
- $\bar{\mathbf{X}} = \sum \alpha_i \mathbf{\Gamma}_i$ - imperfection
- α_i - weights
- $\mathbf{\Gamma}_i$ - base functions (eigenvectors, etc..)
- λ_L - ultimate load (not necessary critical load)

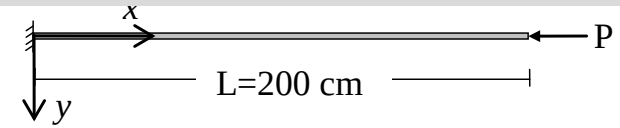
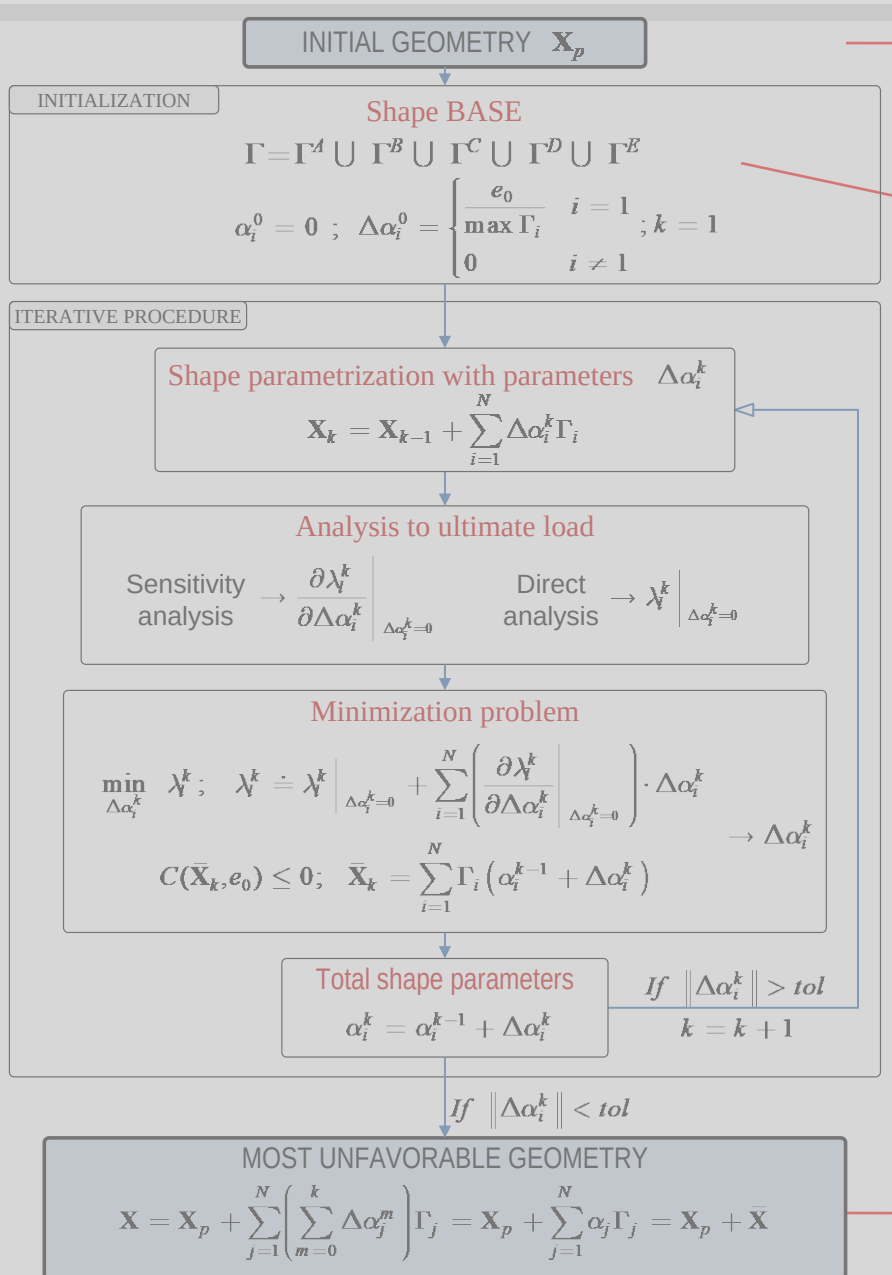
Problem: find α_i for which structure exhibit minimum λ_L

For gradient based optimisation we need gradient: $\frac{\partial \lambda_L}{\partial \alpha_i}$

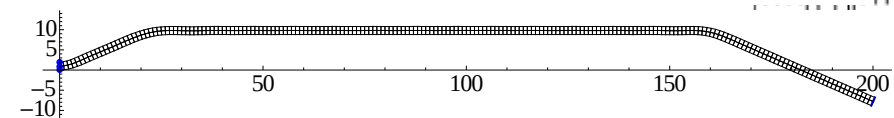
Design velocity field of imperfect geometry: $\frac{\partial \mathbf{X}(\alpha_i)}{\partial \alpha_i} = \mathbf{\Gamma}_i$

Kristanic, Korelc Comput Mech, 2008

The evaluation of the most unfavorable imperfection



Most unfavorable imperfection from the combination of 10 base shapes;
 $e_0=1$, Scale factor=10:



The evaluation of the most unfavorable imperfection

INITIAL GEOMETRY $\mathbf{X}_p$

INITIALIZATION

Shape BASE

$$\Gamma = \Gamma^A \cup \Gamma^B \cup \Gamma^C \cup \Gamma^D \cup \Gamma^E$$

$$\alpha_i^0 = 0 ; \Delta\alpha_i^0 = \begin{cases} \frac{e_0}{\max \Gamma_i} & i = 1 \\ 0 & i \neq 1 \end{cases} ; k = 1$$

$$\Gamma = \Gamma^A \cup \Gamma^B \cup \Gamma^C \cup \Gamma^D \cup \Gamma^E$$

ITERATIVE PROCEDURE

Shape parametrization with parameters $\Delta\alpha_i^k$

$$\mathbf{X}_k = \mathbf{X}_{k-1} + \sum_{i=1}^N \Delta\alpha_i^k \Gamma_i$$

Analysis to ultimate load

$$\text{Sensitivity analysis} \rightarrow \left. \frac{\partial \lambda_V^k}{\partial \Delta\alpha_i^k} \right|_{\Delta\alpha_i^k=0} \quad \text{Direct analysis} \rightarrow \lambda_V^k \Big|_{\Delta\alpha_i^k=0}$$

Minimization problem

$$\min_{\Delta\alpha_i^k} \lambda_V^k ; \lambda_V^k = \lambda_V^k \Big|_{\Delta\alpha_i^k=0} + \sum_{i=1}^N \left(\left. \frac{\partial \lambda_V^k}{\partial \Delta\alpha_i^k} \right|_{\Delta\alpha_i^k=0} \right) \cdot \Delta\alpha_i^k \rightarrow \Delta\alpha_i^k$$

$$C(\bar{\mathbf{X}}_k, e_0) \leq 0 ; \bar{\mathbf{X}}_k = \sum_{i=1}^N \Gamma_i (\alpha_i^{k-1} + \Delta\alpha_i^k)$$

Total shape parameters

$$\alpha_i^k = \alpha_i^{k-1} + \Delta\alpha_i^k$$

$$\text{If } \|\Delta\alpha_i^k\| > tol \\ k = k + 1$$

$$\text{If } \|\Delta\alpha_i^k\| < tol$$

MOST UNFAVORABLE GEOMETRY

$$\mathbf{X} = \mathbf{X}_p + \sum_{j=1}^N \left(\sum_{m=0}^k \Delta\alpha_j^m \right) \Gamma_j = \mathbf{X}_p + \sum_{j=1}^N \alpha_j \Gamma_j = \mathbf{X}_p + \bar{\mathbf{X}}$$

A – buckling shapes

B – eigenvectors of $\mathbf{K}_L$

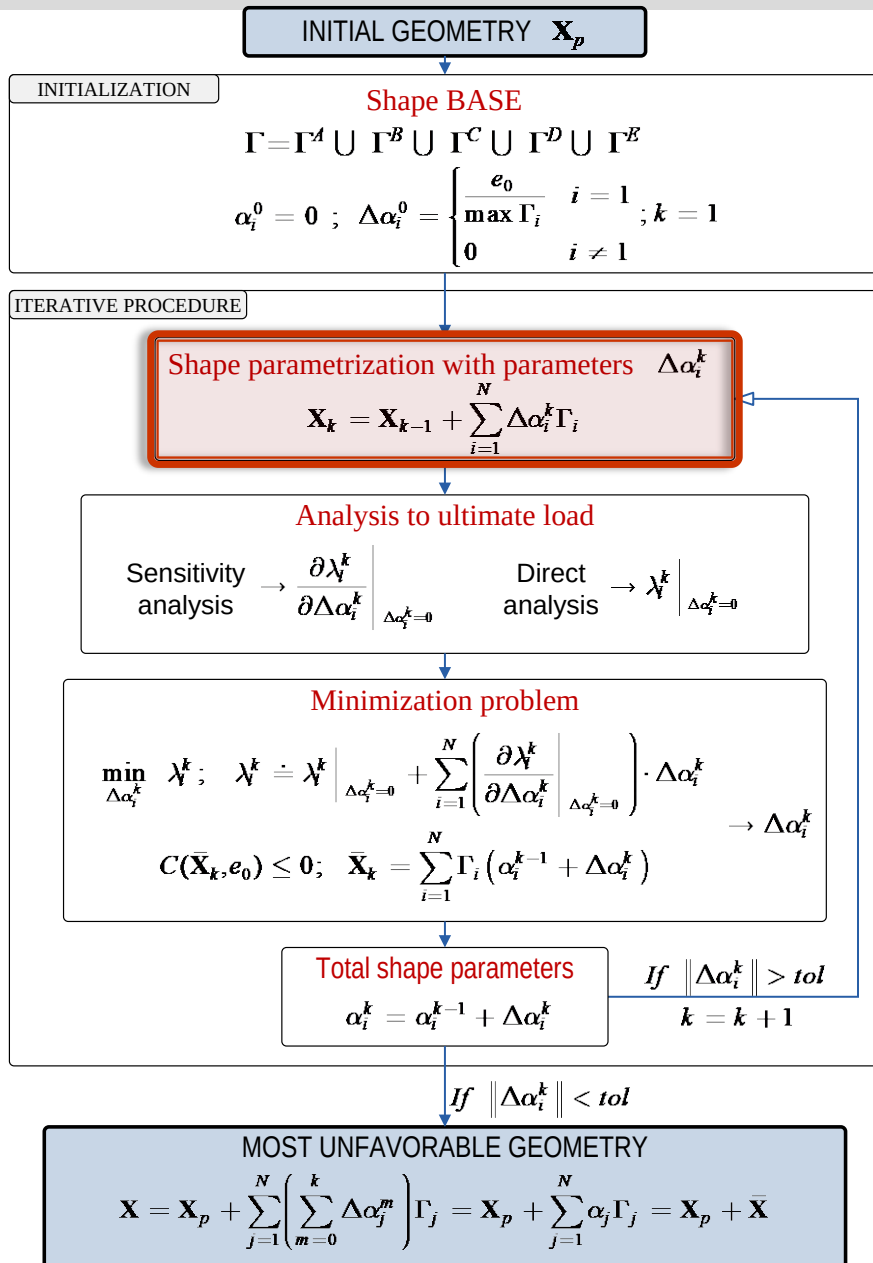
C – eigenvectors of $\bar{\mathbf{K}}$ (alternative boundary conditions)

D – deformation shapes

E – empirically known unfavorable shapes



The evaluation of the most unfavorable imperfection



SHAPE PARAMETRIZATION
with parameters

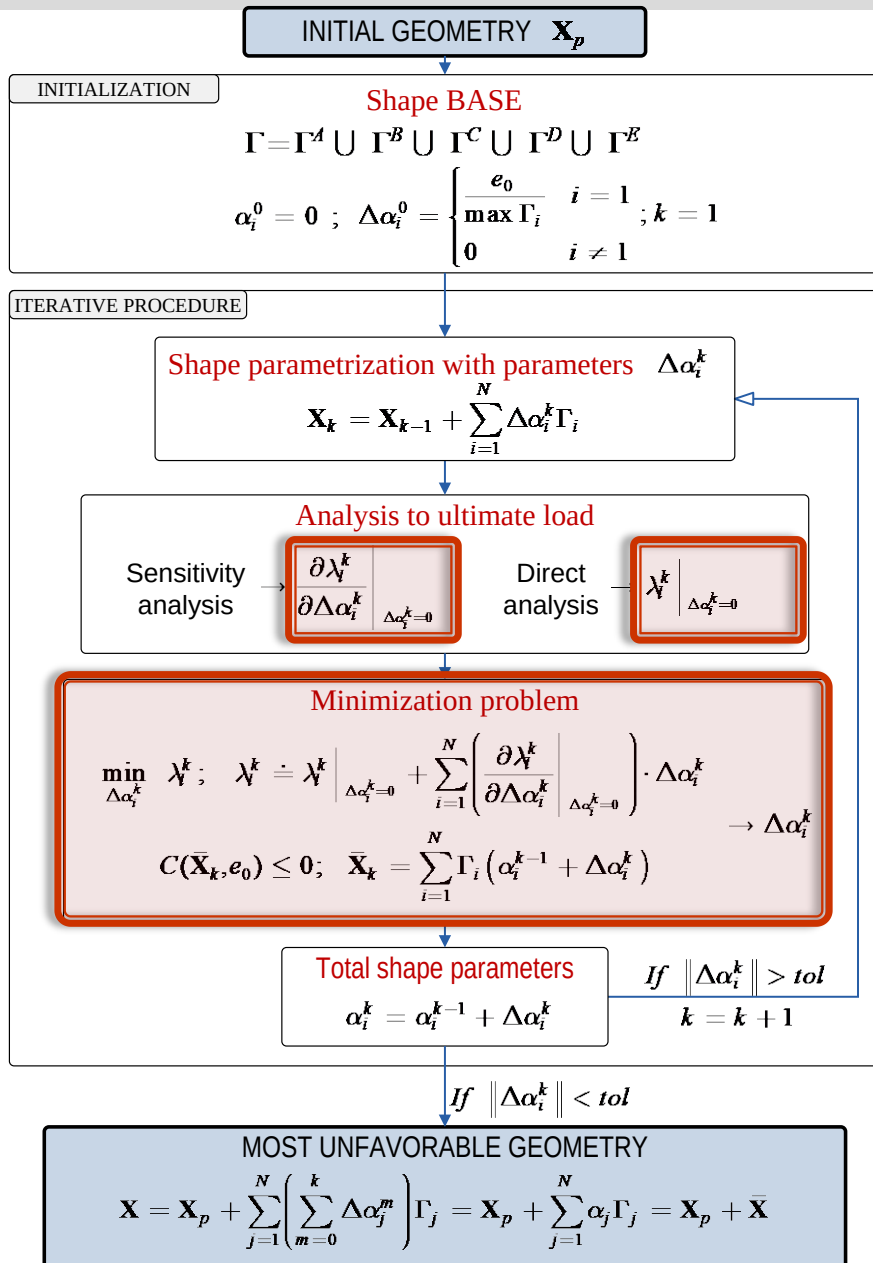
$$\Delta\alpha_j^k$$

Structure geometry:

$$\mathbf{X}_k = \mathbf{X}_{k-1} + \sum_{i=1}^N \Delta\alpha_i^k \Gamma_i$$



The evaluation of the most unfavorable imperfection



Minimization problem:

$$\min_{\Delta\alpha_i^k} \lambda_l^k$$

$$C(\bar{\mathbf{X}}_k, e_0) \leq 0;$$

- sequential linear programming

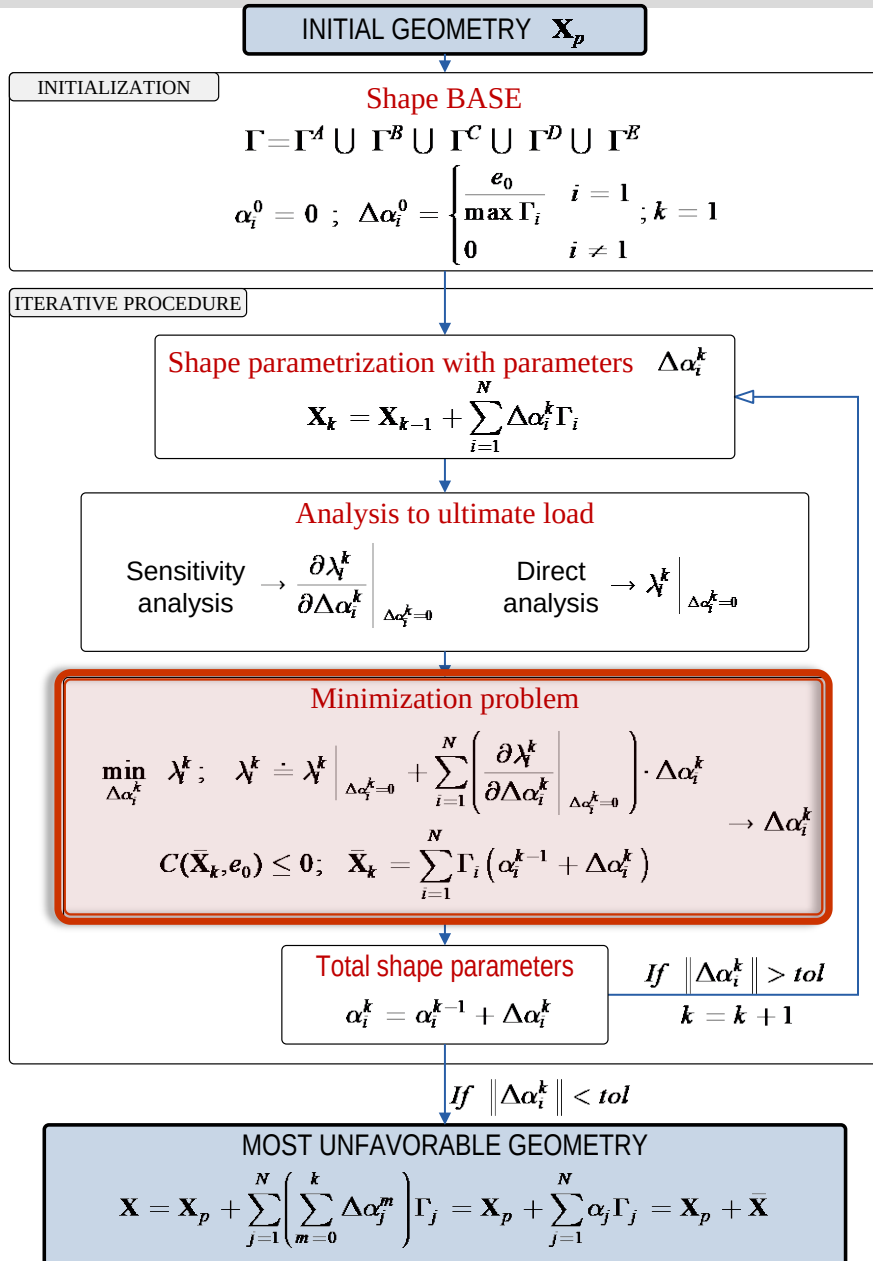
$$\lambda_l^k \doteq \lambda_l^k \big|_{\Delta\alpha_i^k=0} + \sum_{i=1}^N \left(\left. \frac{\partial \lambda_l^k}{\partial \Delta\alpha_i^k} \right|_{\Delta\alpha_i^k=0} \right) \cdot \Delta\alpha_i^k$$

$$\bar{\mathbf{X}}_k = \sum_{i=1}^N \Gamma_i (\alpha_i^{k-1} + \Delta\alpha_i^k)$$

$$\alpha_j^k = \alpha_j^{k-1} + \Delta\alpha_j^k$$



The evaluation of the most unfavorable imperfection



Minimization problem:

$$\lambda_l^k \doteq \lambda_l^k \Big|_{\Delta\alpha_i^k=0} + \sum_{i=1}^N \left(\frac{\partial \lambda_l^k}{\partial \Delta\alpha_i^k} \Big|_{\Delta\alpha_i^k=0} \right) \cdot \Delta\alpha_i^k$$

- shape sensitivity analysis

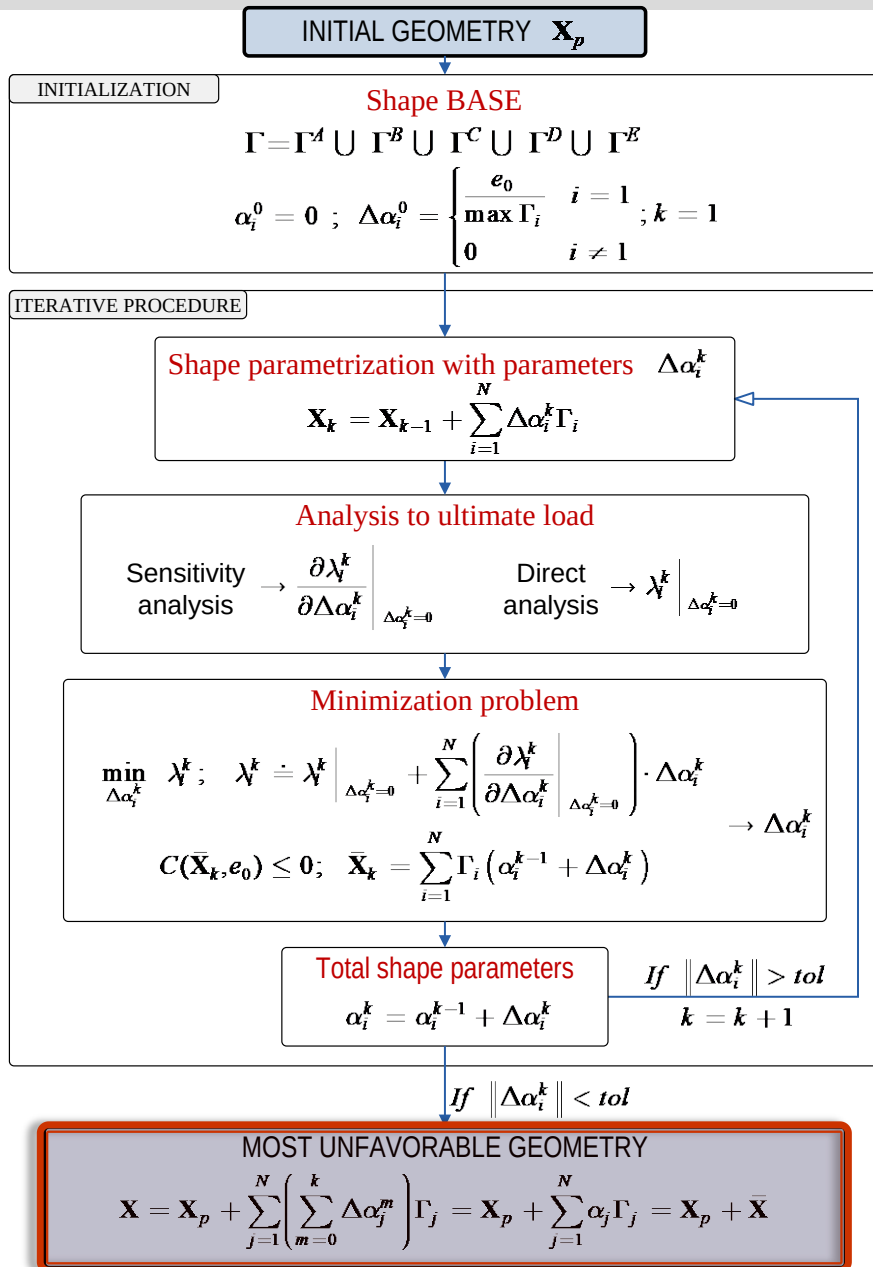
- design velocity field $\frac{\partial \mathbf{X}}{\partial \Delta\alpha_i^k}$

$$\mathbf{X} = \mathbf{X}_p + \sum_{i=1}^N \Gamma_i (\alpha_i^{k-1} + \Delta\alpha_i^k)$$

$$\frac{\partial \mathbf{X}}{\partial \Delta\alpha_i^k} = \frac{\partial \left(\mathbf{X}_p + \sum_{i=1}^N \Gamma_i (\alpha_i^{k-1} + \Delta\alpha_i^k) \right)}{\partial \Delta\alpha_i^k} = \delta_{ij} \cdot \Gamma_j = \Gamma_i$$



The evaluation of the most unfavorable imperfection



FINAL GEOMETRY:

$$\mathbf{X} =$$

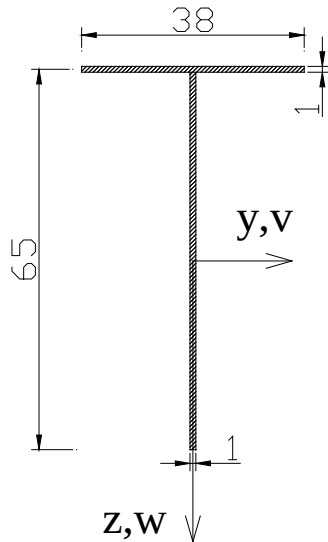
$$= \mathbf{X}_p + \sum_{j=1}^N \left(\sum_{m=0}^k \Delta\alpha_j^m \right) \Gamma_j =$$

$$\mathbf{X}_p + \sum_{j=1}^N \alpha_j \Gamma_j = \mathbf{X}_p + \bar{\mathbf{X}}$$

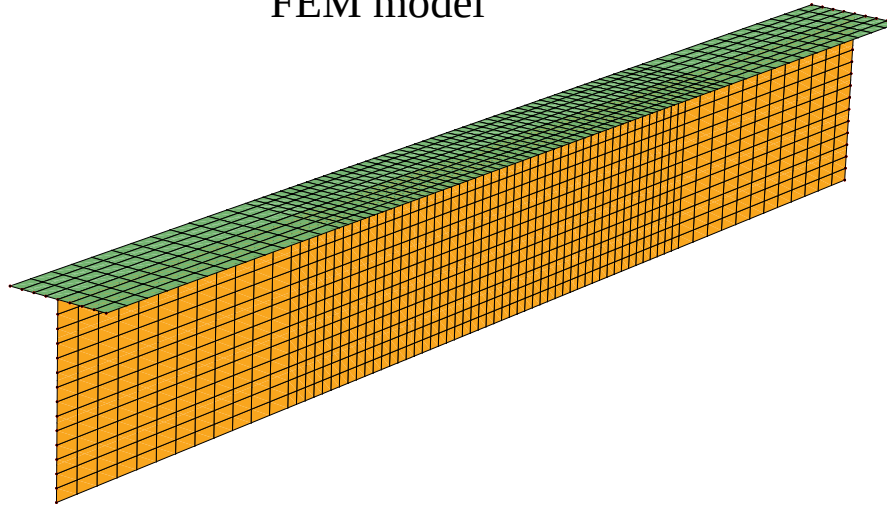


Example: 3D thin-walled T beam

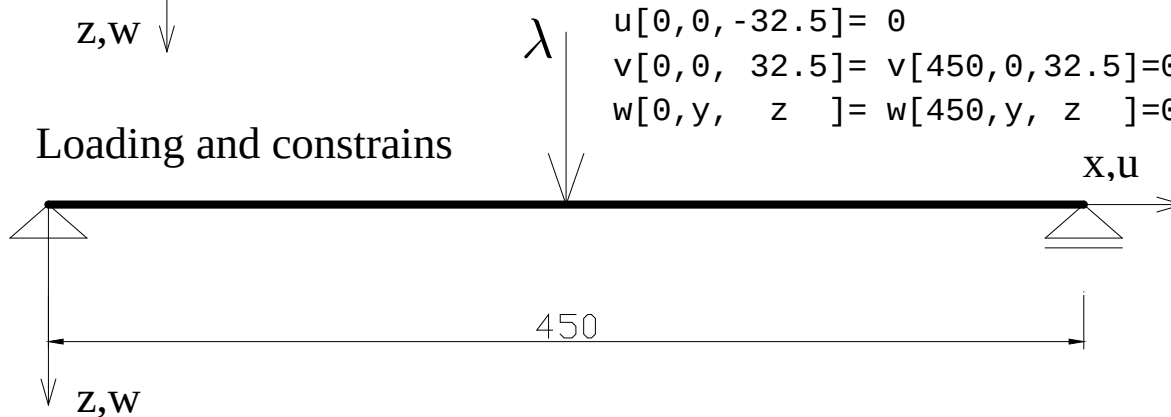
Cross section



FEM model

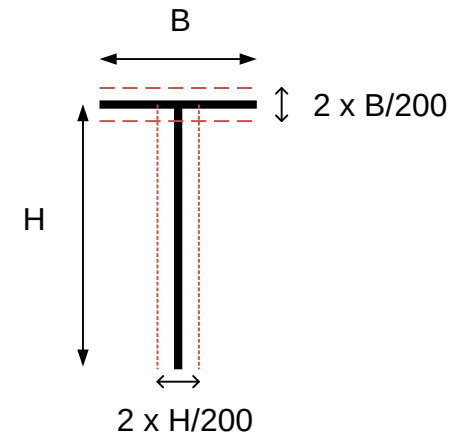


Loading and constraints



$$\begin{aligned} u[0, 0, -32.5] &= 0 \\ v[0, 0, 32.5] &= v[450, 0, 32.5] = 0 \\ w[0, y, z] &= w[450, y, z] = 0 \end{aligned}$$

Imperfection constraints

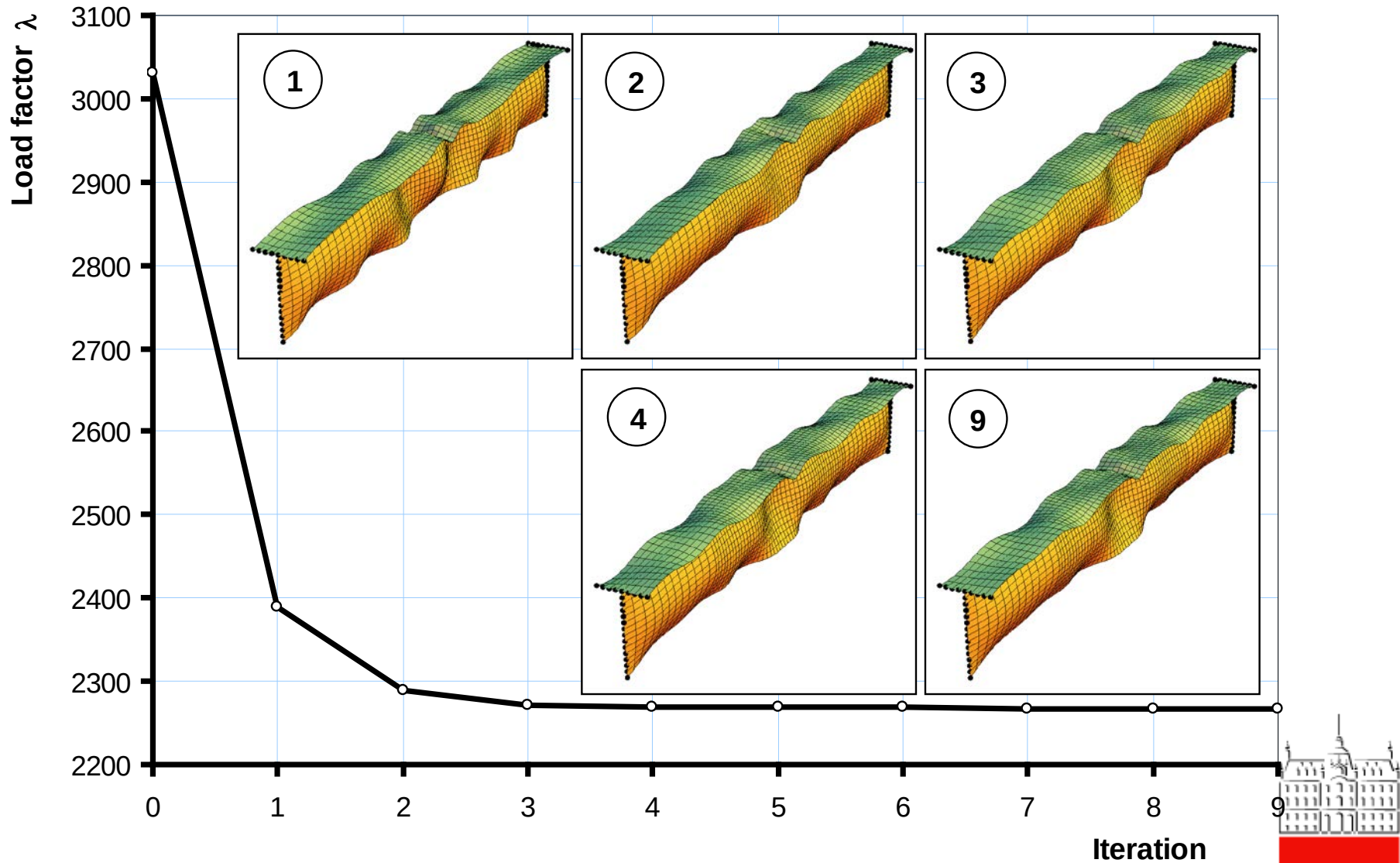


Model:

- 4-node shell elements, 6-parameter shell theory (+2 enhanced modes; warping and in-plane twist parameters), ANS, (K. Wisniewski, 2001)
- Elasto-plastic material model with hardening
- Shape sensitivity

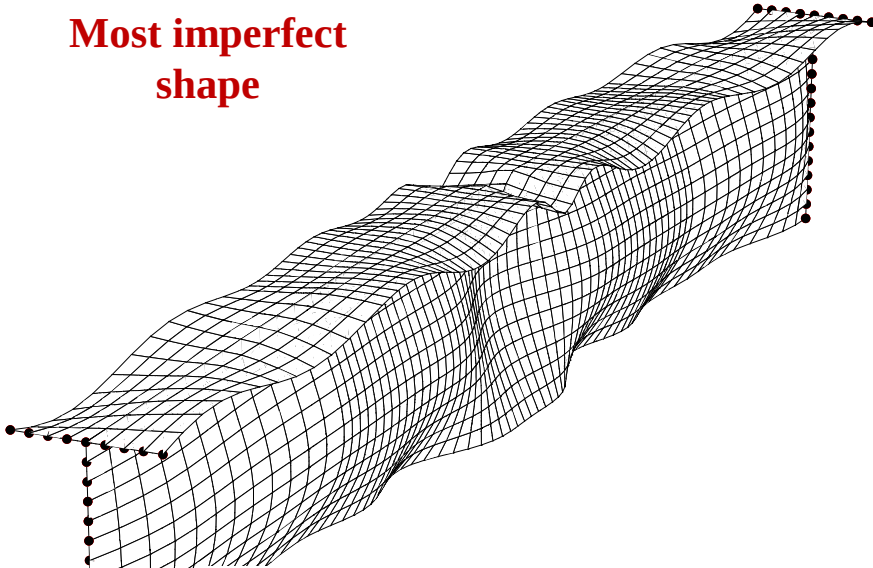


3D thin-walled T beam: ITERATION PROCEDURE

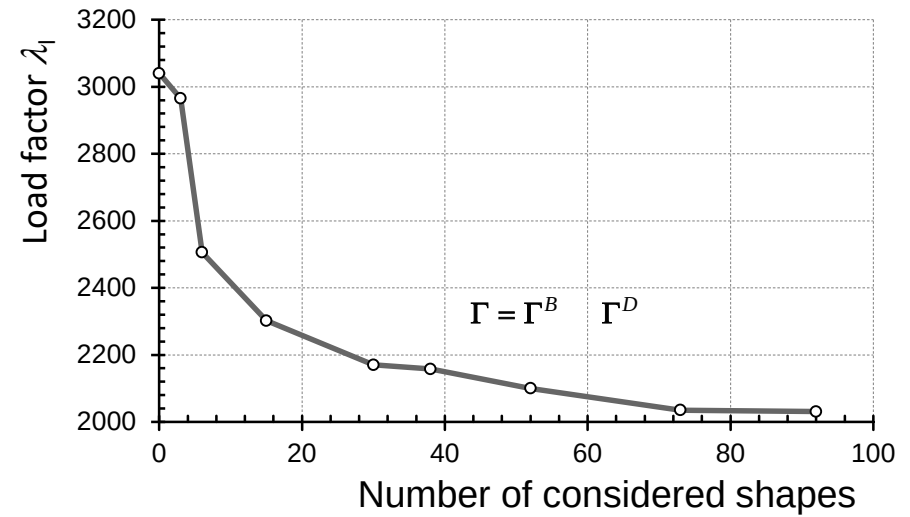
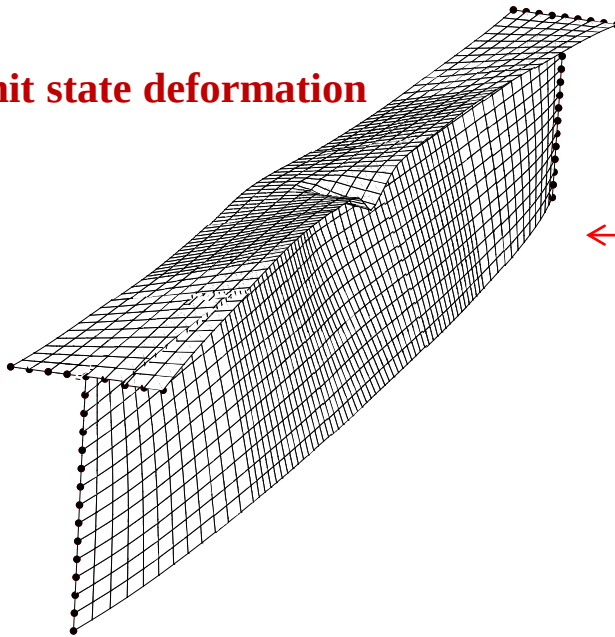


3D thin-walled T beam: RESULTS

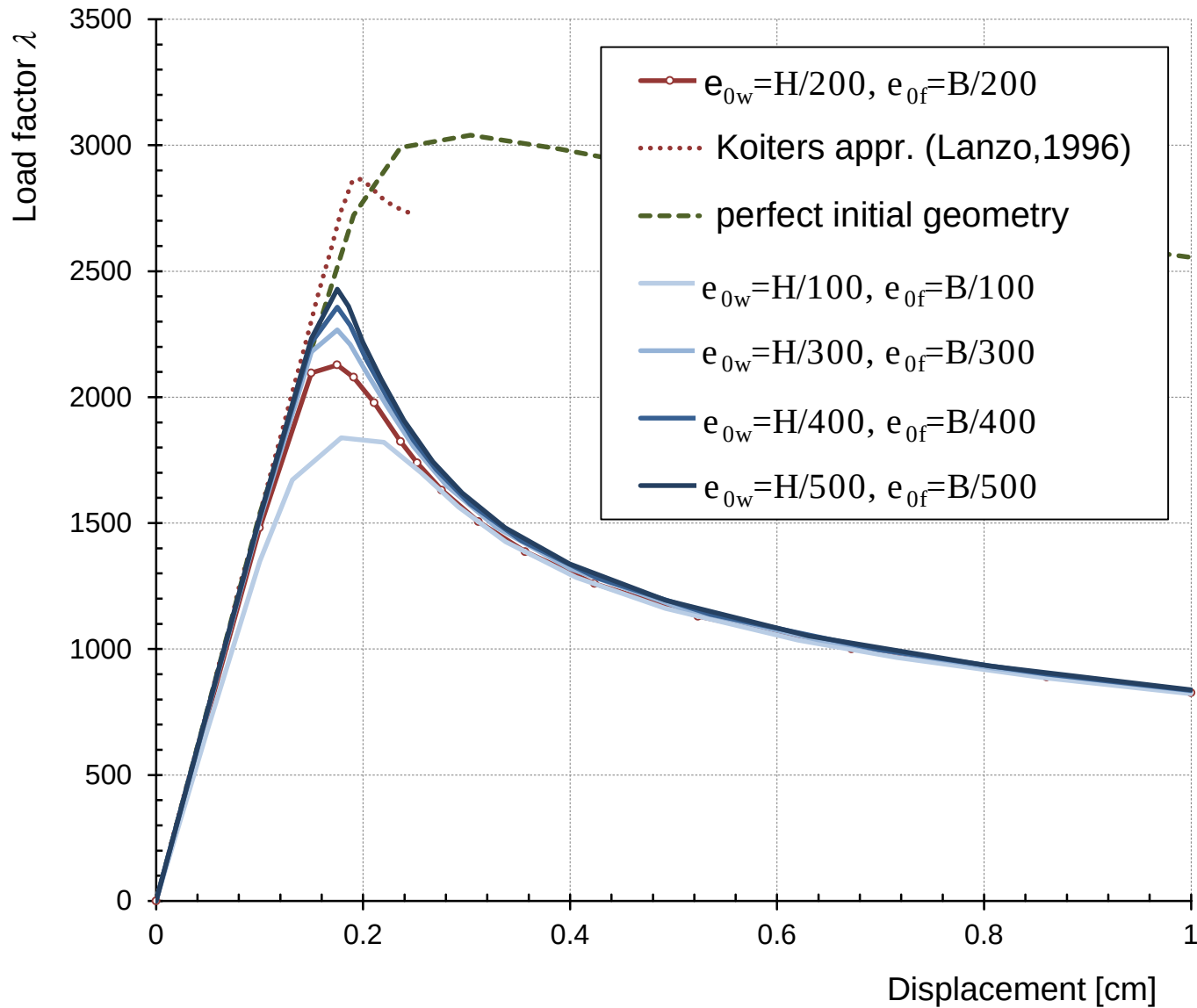
**Most imperfect
shape**



Limit state deformation

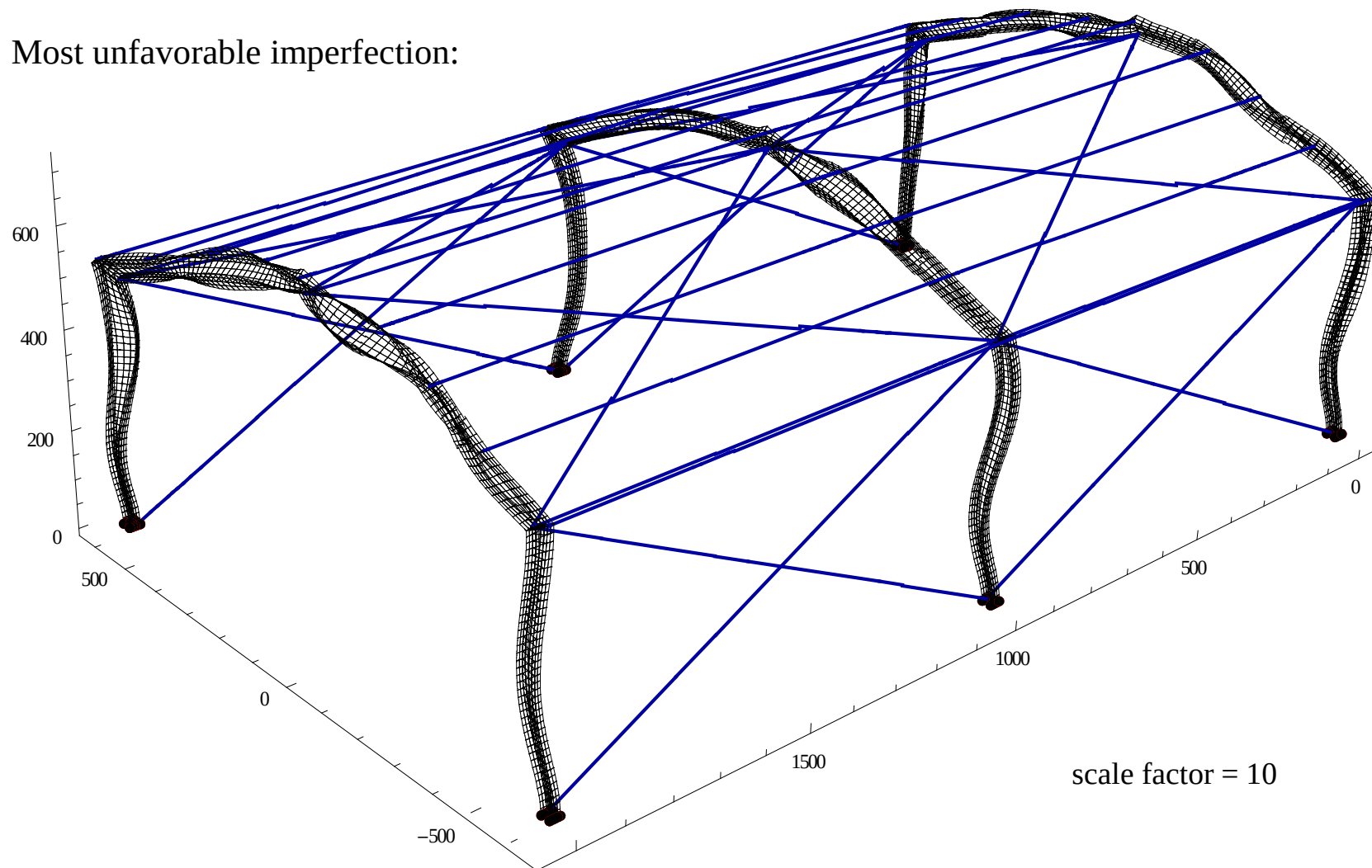


3D thin-walled T beam: COMPARISON

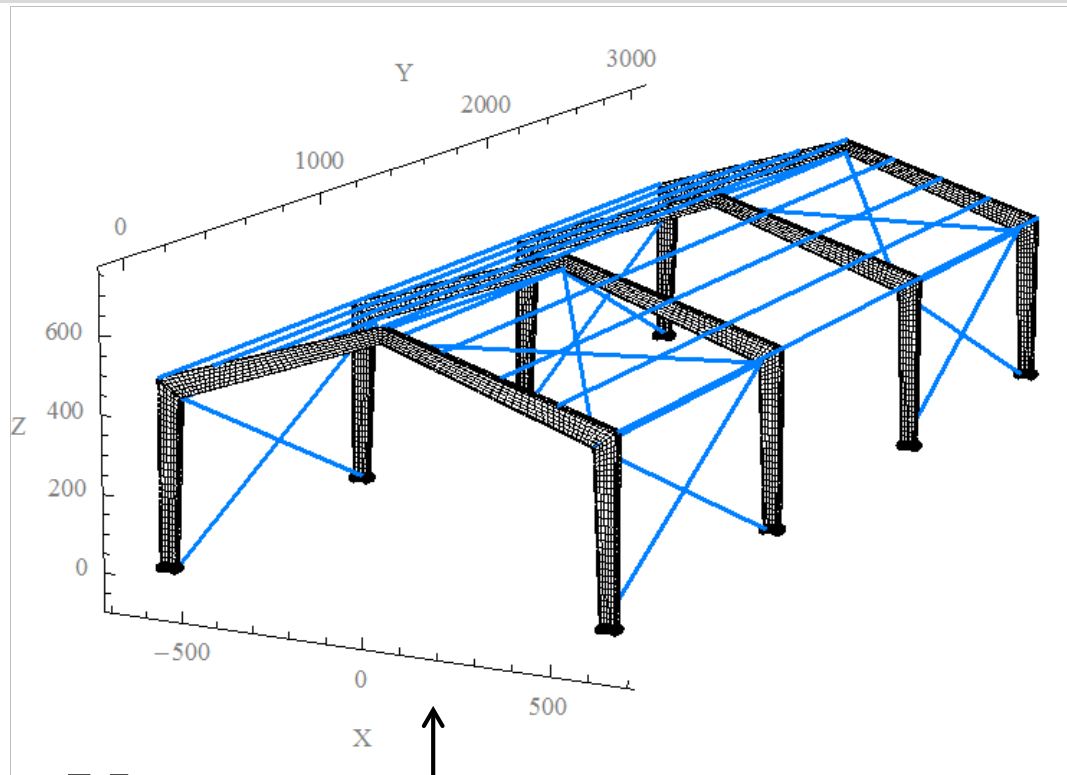


Example: Single story building

Most unfavorable imperfection:



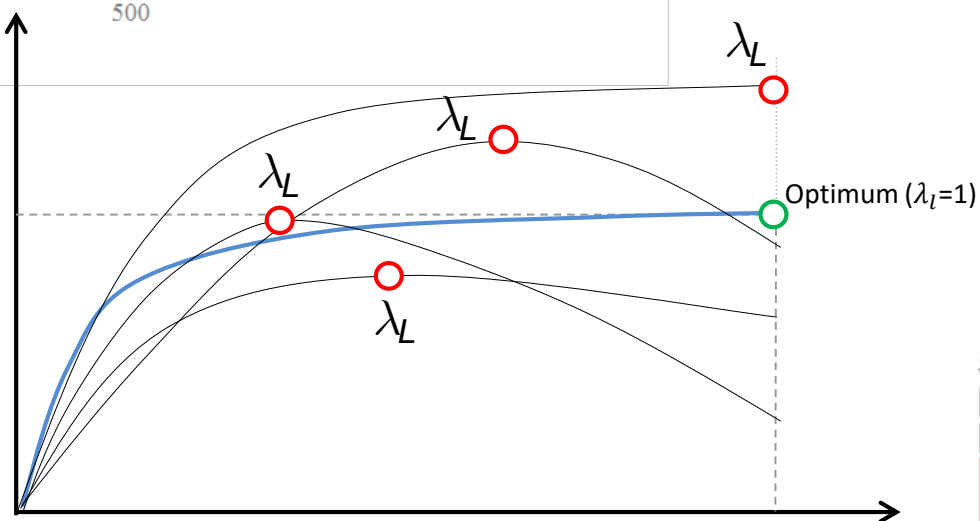
Limit load optimization



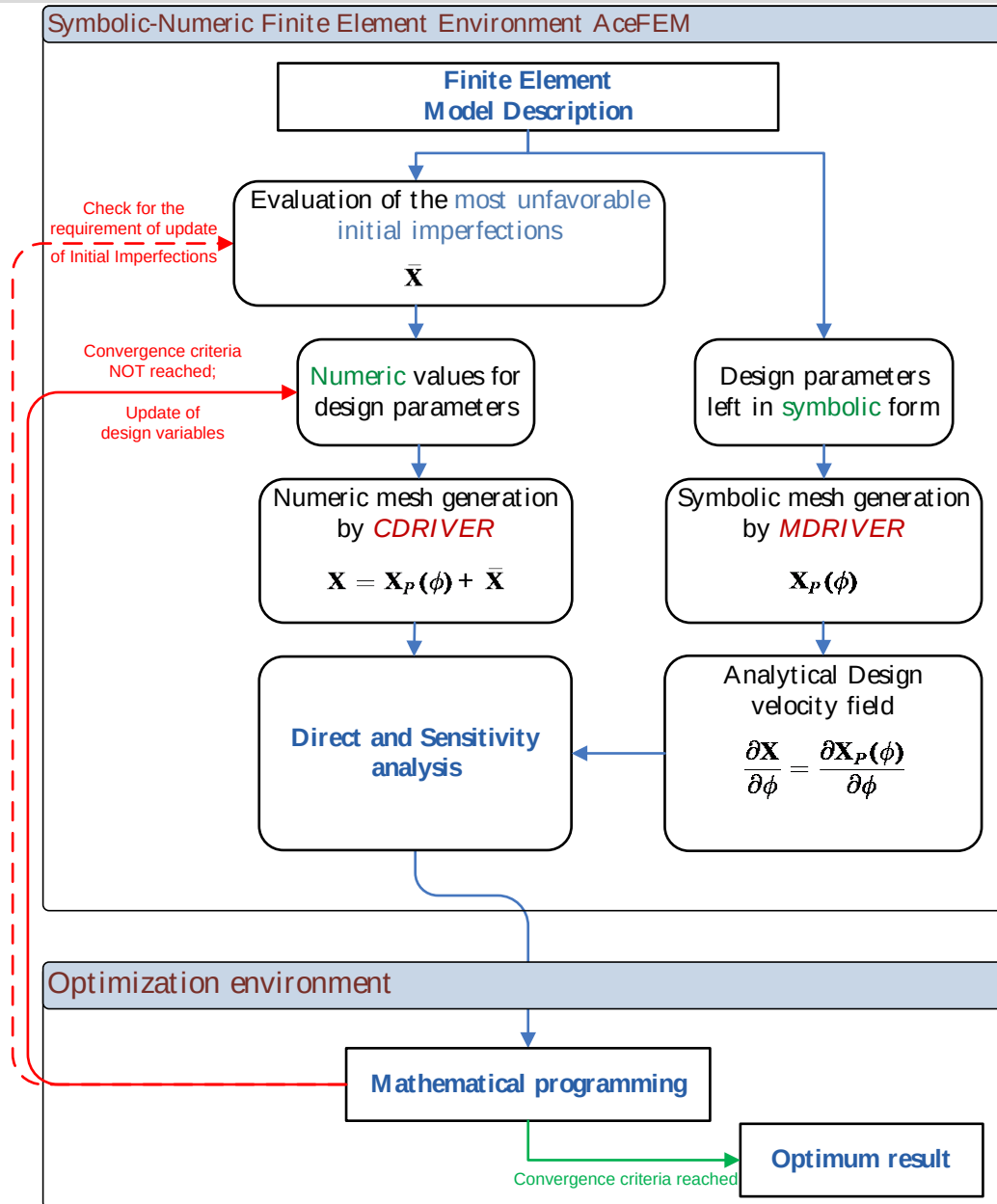
$$\min_{\alpha} V$$

$$\lambda_l = 1$$

$$C(\alpha) \leq 0$$



Algorithm for limit load optimization

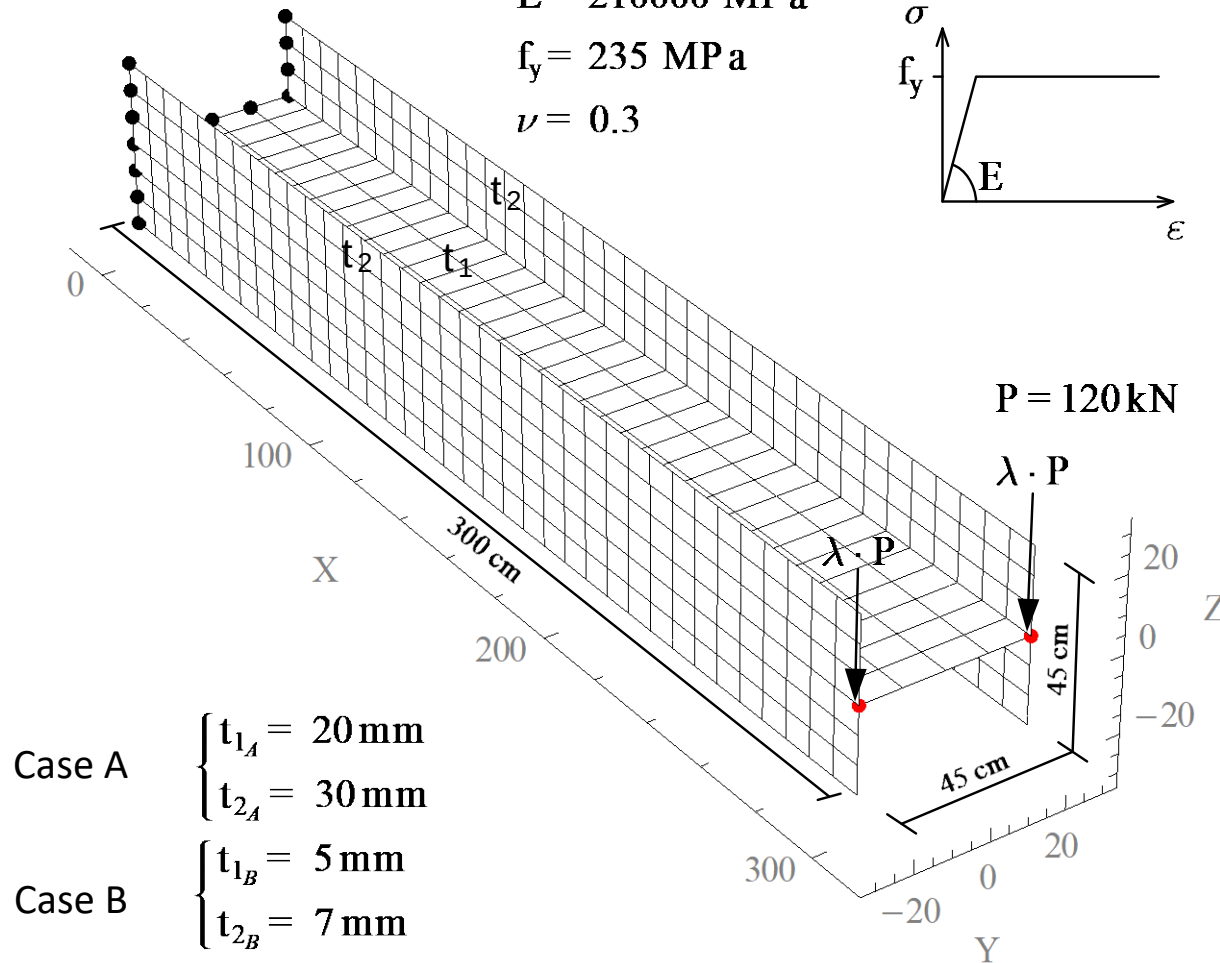
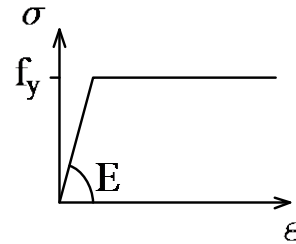


Optimization example - H beam

$$E = 210000 \text{ MPa}$$

$$f_y = 235 \text{ MPa}$$

$$\nu = 0.3$$



Formulation:

$$\min f;$$

$$f = \mathbf{V}(\phi)$$

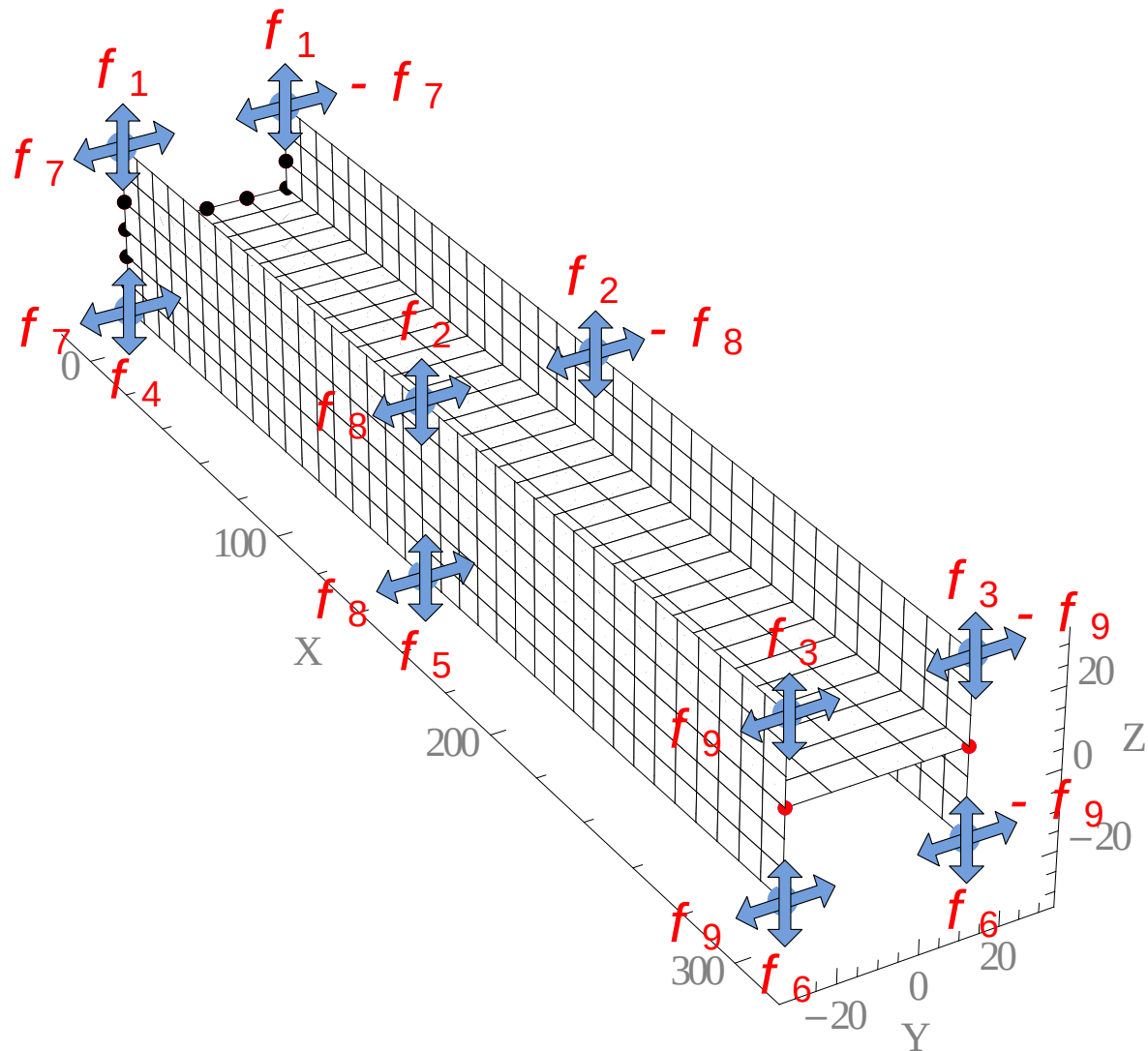
$$\lambda_L(\phi) - 1 = 0$$

$$\Phi_P^k \leq 0$$

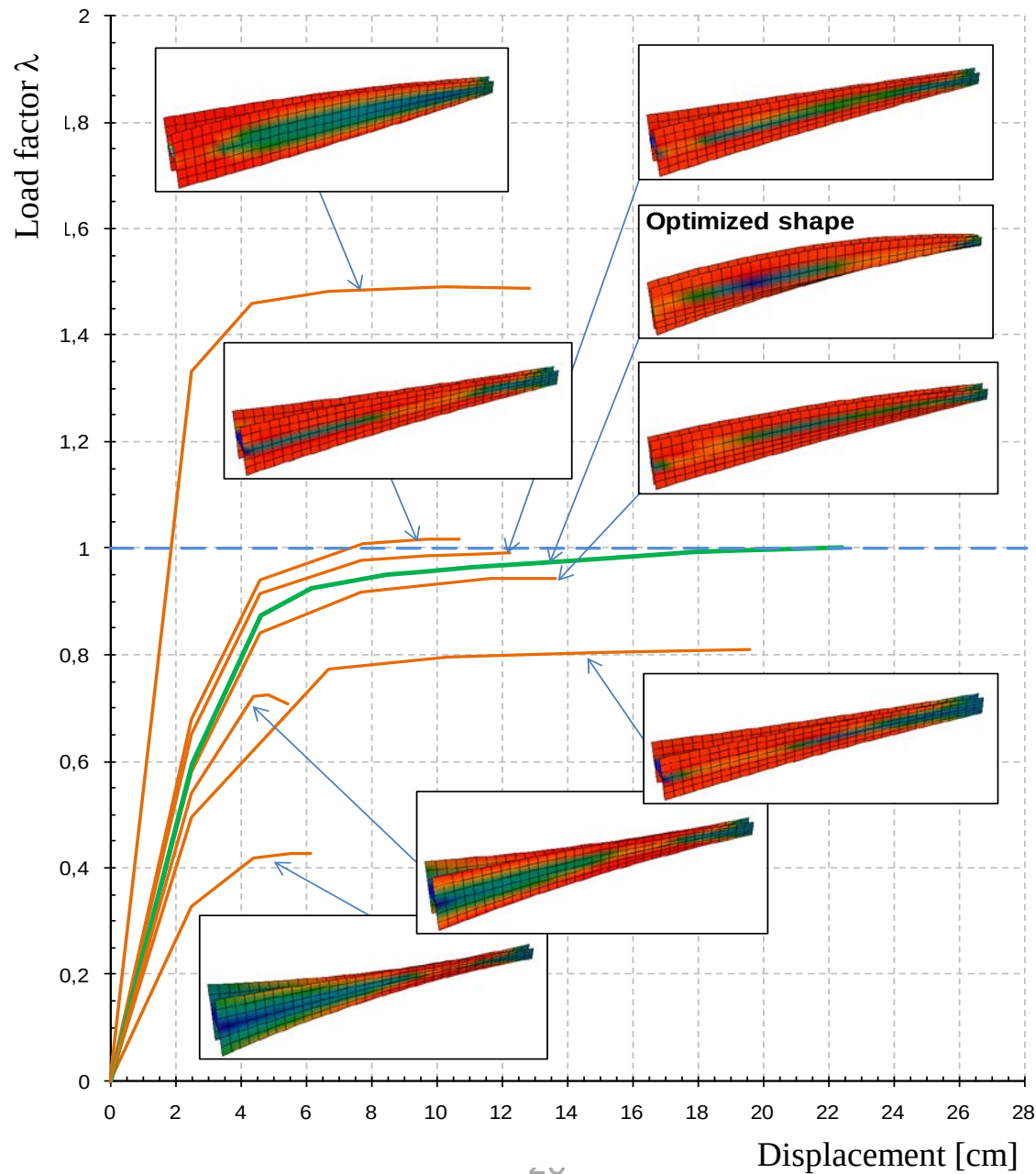
$$f_B = w_1 \mathbf{V} + w_2 \text{Log}(\lambda_L - 1) + w_3 \sum_k \text{Log}(\phi^k - \phi_B^k)$$



Optimization example - H beam

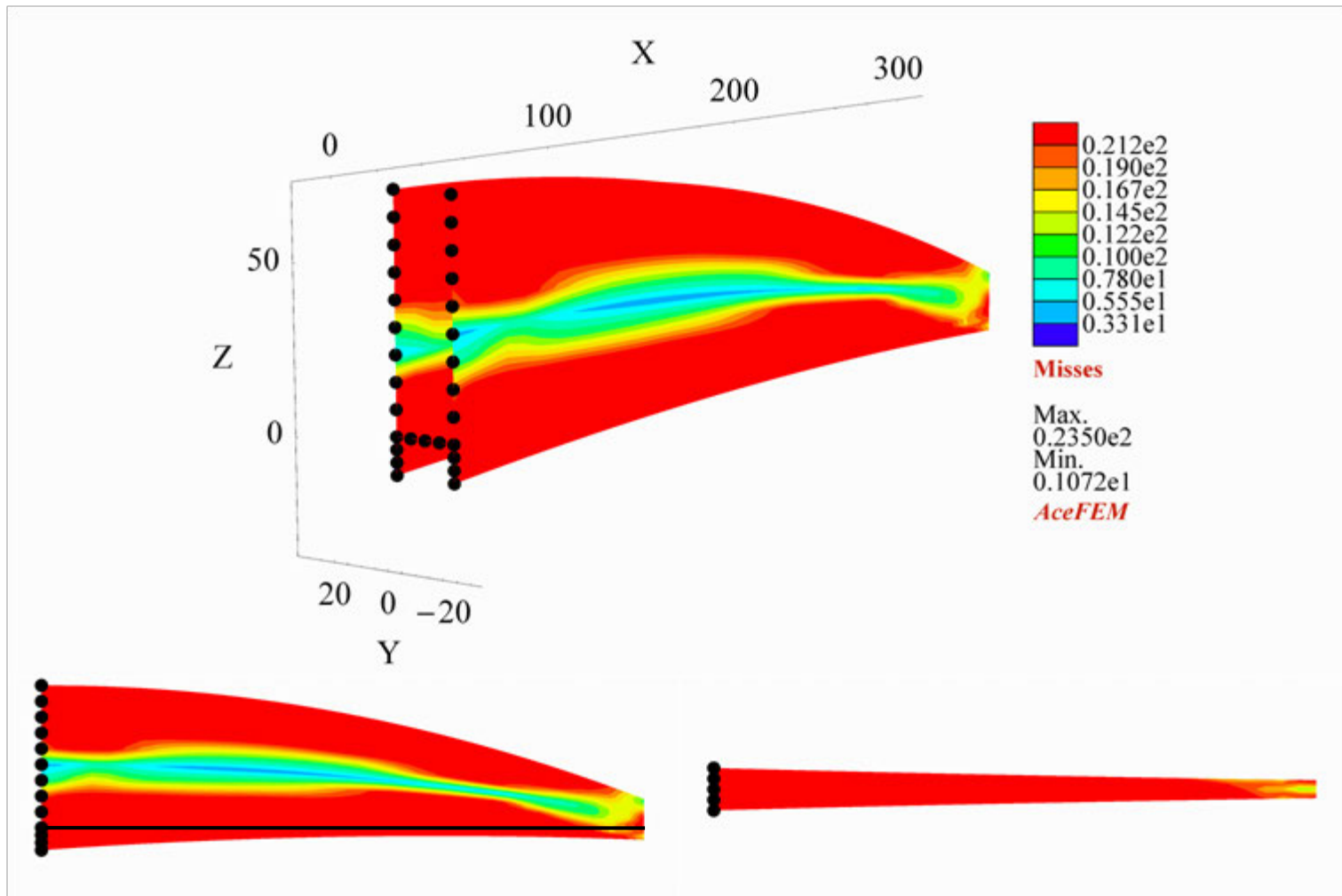


Optimization example - H beam



Optimization example - H beam

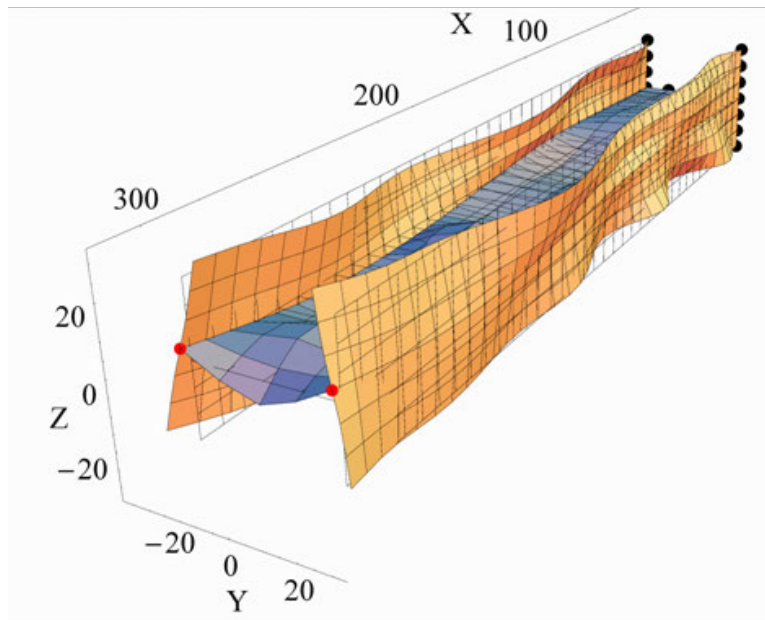
Mises stress at limit load for the optimal



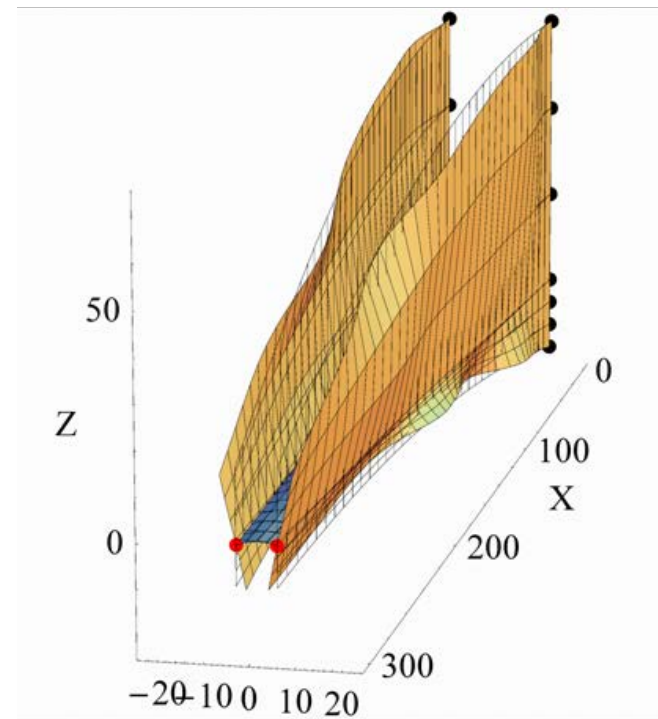
Optimization example - H beam

Most unfavorable imperfections:

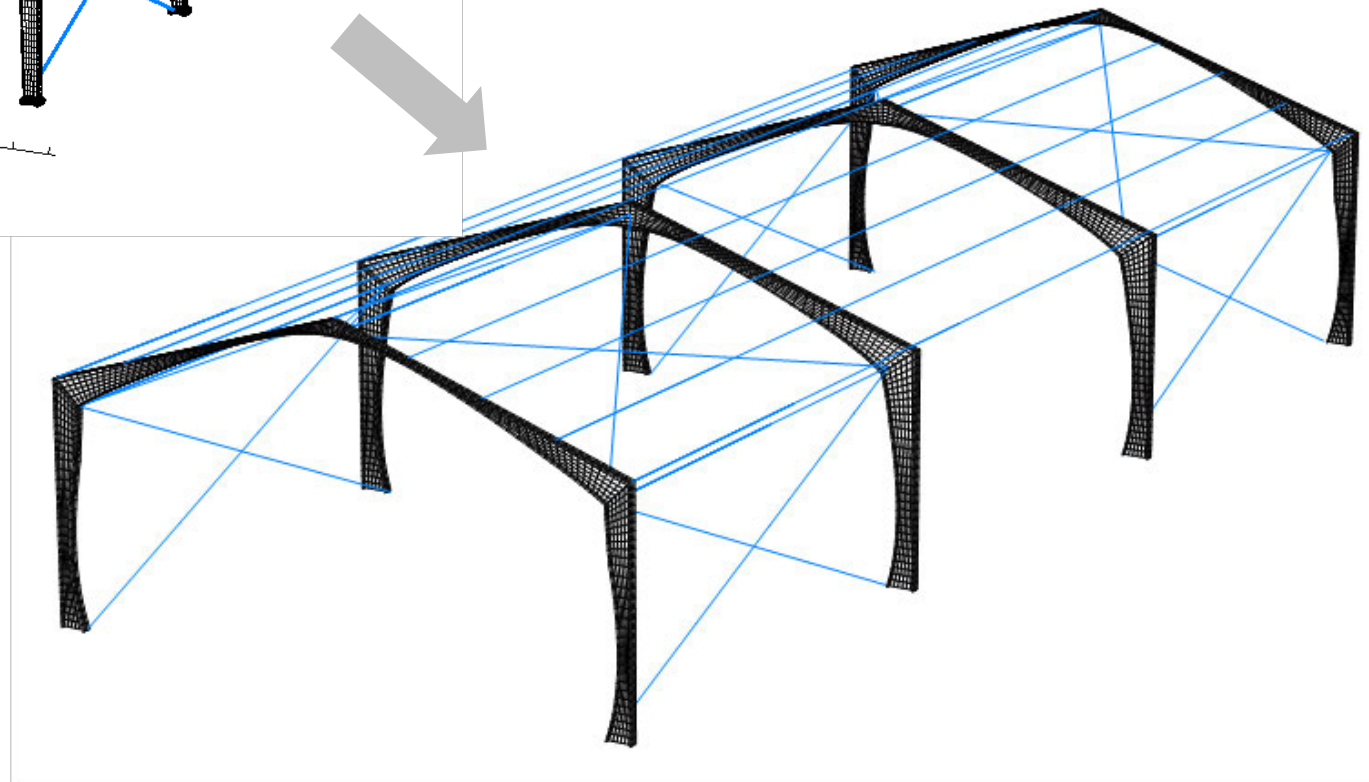
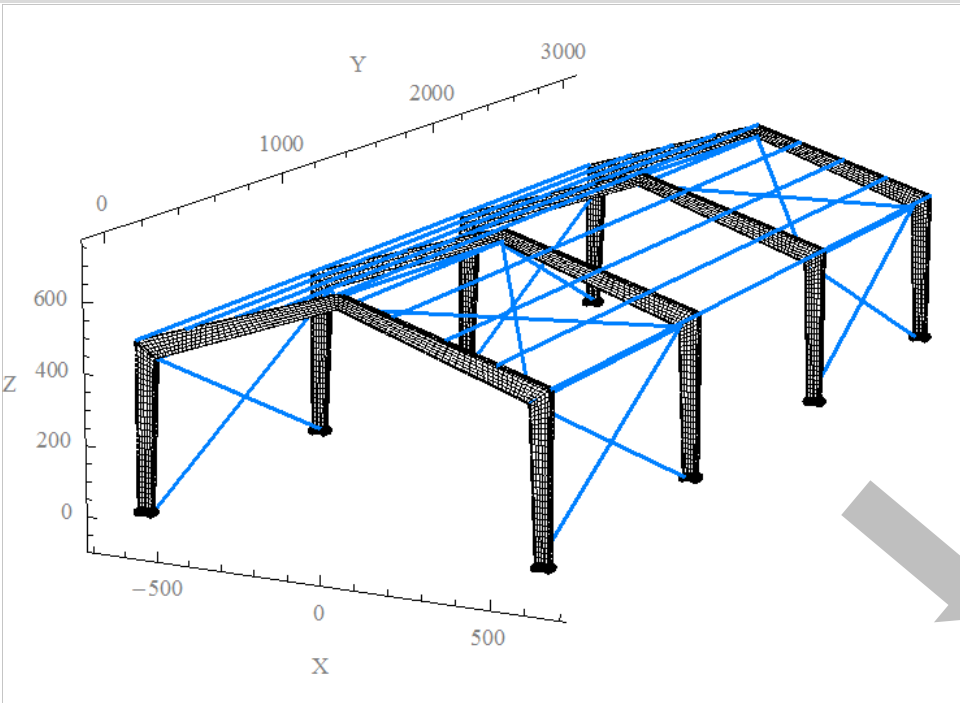
Initial shape:



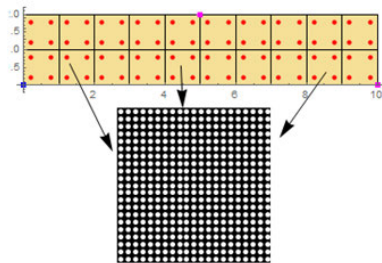
Optimal shape:



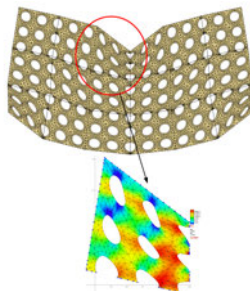
Optimization example - frame



Multi-scale modeling and sensitivity analysis



FE^2

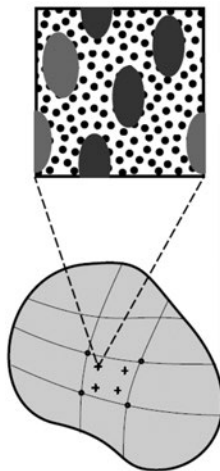


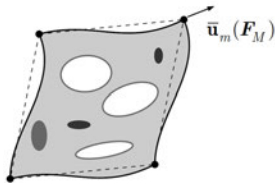
mesh-in-element (MIEL)

- two level finite element homogenization approach FE^2 is appropriate for the problem where scales are weakly coupled
- when the scales are strongly coupled or in the region of high gradients the mesh-in-element or MIEL scheme can be applied

Finite strain FE² computational framework

- **Micro problem:** micro FE mesh
 - **input:** macro deformation gradient $\mathbf{F}_M$
 - **load:** essential boundary conditions
 - **calculate:** $\mathbf{P}_M = \{\mathbf{P}_m\}$ - average stress tensor over volume (micro response)
 - **calculate:** $\mathbf{A}_M = \{\frac{\partial \mathbf{P}_m}{\partial \mathbf{F}_M}\}$ - average tangent
- **Macro problem:** macro FE mesh
 - **input:** integration points $\mathbf{P}_M, \mathbf{A}_M$
 - **calculate:** element weak form
$$\mathbf{R}_M = \int \mathbf{P}_M : \frac{\partial \mathbf{F}_M}{\partial \mathbf{u}_e} d\Omega$$
 - element tangent
$$\mathbf{K}_M = \int \frac{\partial \mathbf{R}_M}{\partial \mathbf{u}_e} \Big|_{\frac{\partial \mathbf{P}_M}{\partial \mathbf{F}_M} = \mathbf{A}_M}$$
 - **output:** new $\mathbf{F}_M$





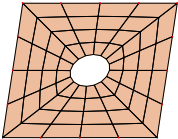
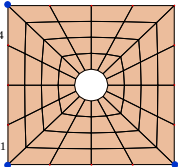
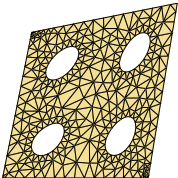
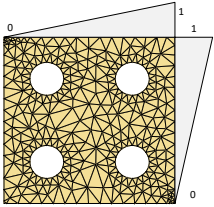
Standard approach

- $\bar{\mathbf{u}}_m = (\mathbf{F}_M - \mathbf{I}) \mathbf{X}_m$ essential boundary conditions of micro problem
- micro problem depends on 9 components of asymmetric $\mathbf{F}_M$
 $\Phi = \{F_{M,11}, F_{M,12}, F_{M,13}, F_{M,21}, \dots\}$ sensitivity parameters

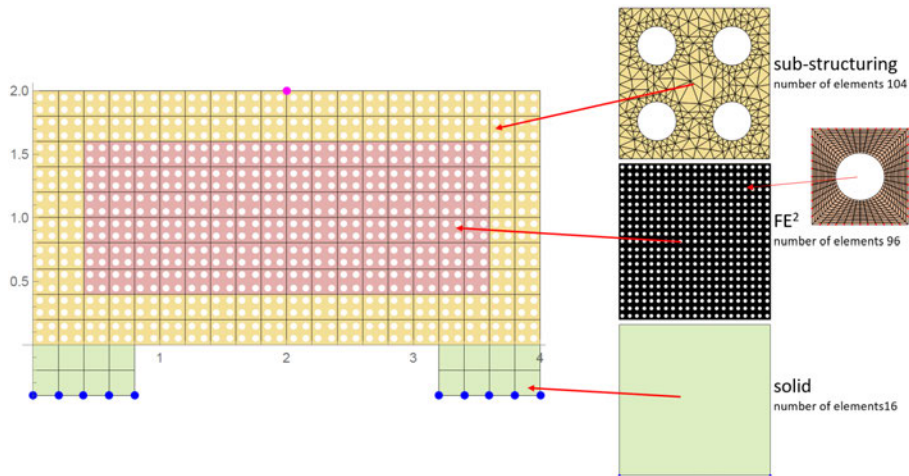
Required quantities

- calculate: $\mathbf{P}_M = \{\mathbf{P}\}$
- calculate: $\mathbf{A}_M = \left\{ \frac{\partial \mathbf{P}_m}{\partial \mathbf{F}_M} \right\} = \left\{ \frac{\partial \mathbf{P}_m}{\partial \mathbf{u}_m} \frac{D\mathbf{u}_m}{D\mathbf{F}_M} \right\}$
- explicit part: $\frac{\partial \mathbf{P}_m}{\partial \mathbf{u}_m}$ - AceGen
- implicit part: derivative $\frac{D\mathbf{u}_m}{D\mathbf{F}_M}$ - first order sensitivity problem with respect to prescribed essential boundary conditions

Velocity fields for FE² and MIEL.

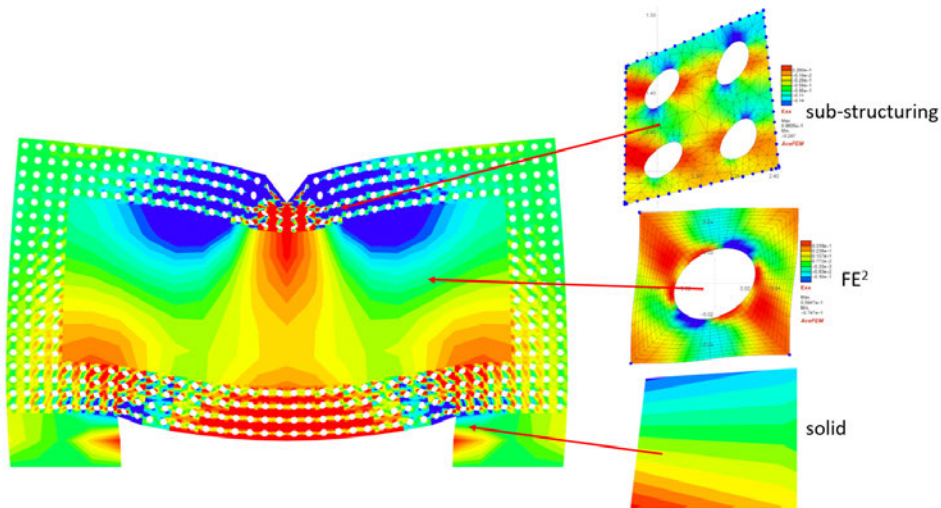
method	micro-scale	characteristic design velocity field
FE ²		$\frac{\partial \bar{u}_4}{\partial F_{11}} = x_4$ $\frac{\partial \bar{u}_1}{\partial F_{11}} = x_1$ $\frac{\partial \bar{u}_2}{\partial F_{11}} = x_2$  $\frac{\partial \bar{u}_{mi}}{\partial F_{Mjk}} = \delta_{ij} X_{mk}$
MIEL		 $\frac{\partial \sum N_i(\Xi) u_{Mi}}{\partial u_{Mj}} = N_j(\Xi)$

Example: Sub-structuring, FE^2 and single-scale in one model



Visualization of results

Results for strains E_{xx}



Convergence properties - sub-structuring + FE² + solid

- Sensitivity calculation in every micro sub-step

elastic	hyperelastic	elasto-plastic	elasto-plastic
0.105	1.05×10^{-2}	4.65×10^{-4}	4.51×10^{-4}
1.41×10^{-15}	1.11×10^{-4}	2.49×10^{-6}	2.12×10^{-6}
/	1.05×10^{-7}	2.61×10^{-8}	2.48×10^{-8}
/	2.03×10^{-14}	3.82×10^{-13}	1.47×10^{-13}

- Sensitivity calculation only at the end of micro solution

elastic	hyperelastic	elasto-plastic	elasto-plastic
0.105	1.05×10^{-2}	6.04×10^{-4}	6.01×10^{-4}
1.41×10^{-15}	1.11×10^{-4}	2.36×10^{-4}	2.18×10^{-4}
/	1.05×10^{-7}	1.14×10^{-4}	9.75×10^{-5}
/	2.03×10^{-14}	1.79×10^{-5}	1.32×10^{-5}
/	/	5.17×10^{-7}	2.01×10^{-7}
/	/	1.88×10^{-9}	3.46×10^{-9}
/	/	2.94×10^{-11}	3.65×10^{-11}
/	/	3.13×10^{-13}	3.67×10^{-13}



Stochastic field - (e.g. thickness of the corroded metal plate)

- $w(\mathbf{X}, \theta)$ - stochastic field over given domain (any parameter)
- discretization: Karhunen-Loève expansion of stochastic field
- $w(\mathbf{X}, \theta) = \bar{w}(\mathbf{X}) + \sum \sqrt{\lambda} f_j(\mathbf{X}) \eta_j(\theta)$
- $\bar{w}(\mathbf{X})$ - expected value , $f_j(\mathbf{X})$ - eigenfunctions, λ - eigenvalues
- $\eta_j(\theta)$ - KL coefficients (uncorrelated random variables)

Problem: evaluation of the statistics of the response

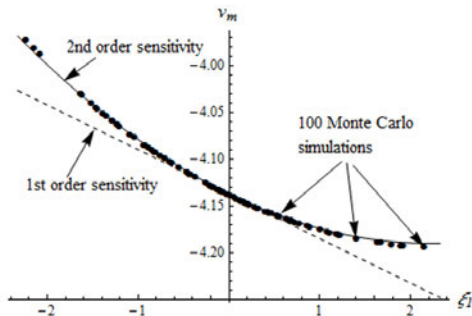
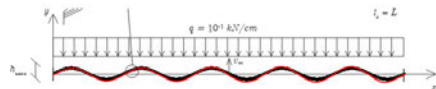
Solution: expansion of the response with respect to $\eta_j(\theta)$

Design velocity field: $\frac{\partial w}{\partial \eta_j} = \sqrt{\lambda} f_j(\mathbf{X})$

Melink, Korelc, Probabilistic Engineering Mechanics, 2014

Example: stochastic analysis of sinusoidal double skin cladding

	Mean (v_m)	Standard deviation (v_m)	Total CPU time
first order sensitivity analysis	-4.1374 cm	0.0555 cm	0.25 s
second order sensitivity analysis	-4.1263 cm	0.0577 cm	0.39 s
second order finite difference method	-4.1263 cm	0.0577 cm	46.58 s
10^3 MC simulations	-4.1262 cm	0.0588 cm	271 s
$4 \cdot 10^4$ MC simulations	-4.1264 cm	0.0578 cm	10084 s



Asymptotic numerical methods

- expansion of the response with respect to parameter of the problem (load level λ)
- $\mathbf{R}(\mathbf{p}(\lambda), \lambda) = \mathbf{0}$ - response
- $\mathbf{p}(\lambda) = \mathbf{p}_0 + \frac{\partial \mathbf{p}}{\partial \lambda} \delta \lambda + 1/2 \frac{\partial^2 \mathbf{p}}{\partial \lambda^2} \delta \lambda^2 \dots$
- improving iterative procedures (Newton-Raphson) by better initial guess

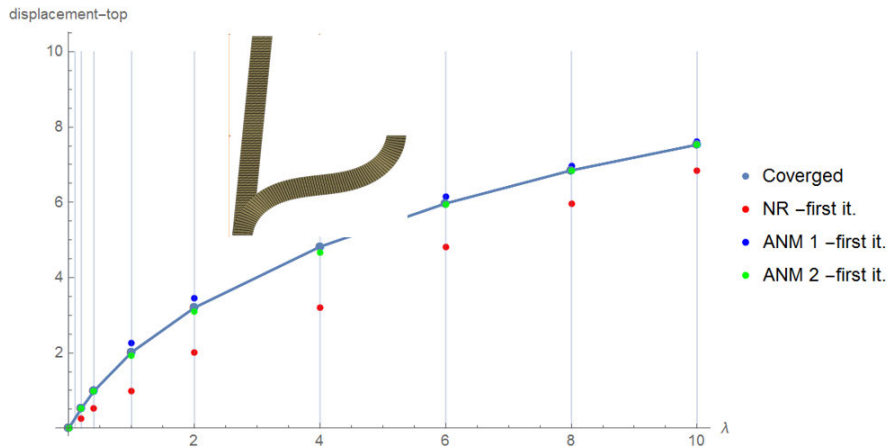
Proportional loading: $\bar{\mathbf{p}} = \lambda \mathbf{p}_{ref}$, $F = \lambda \mathbf{F}_{ref}$

Design velocity field:

$$\frac{\partial \bar{\mathbf{p}}}{\partial \lambda} = \mathbf{p}_{ref}, \quad \frac{\partial^2 \bar{\mathbf{p}}}{\partial \lambda^2} = 0 \qquad \frac{\partial \mathbf{F}}{\partial \lambda} = \mathbf{F}_{ref}, \quad \frac{\partial^2 \mathbf{F}}{\partial \lambda^2} = 0$$

Korelc, Finite Elements in Analysis and Design, 2010

Bending of the column



- the use of sensitivity analysis can lead to a new, more efficient type structural analysis and design
- manual coding is not possible for time-dependent, coupled problems
- sensitivity analysis requires advanced software technologies